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Intuitionistic Fuzzy Set Characteristic Analysis with Lexicographical Vector Lattice Structure

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ABSTRACT

The similarity measure design for the different patterns has been proposed based on the information characteristic analysis and the distance measure. Analysis of fuzzy sets (FSs), intuitionistic fuzzy sets (IFSs), and Pythagorean fuzzy sets (PFSs) and characteristics comparison between them also have been carried out with membership and non-membership degrees. Each degree of FSs, IFSs and PFSs is illustrated through figures, and its characteristics are analyzed. The existing similarity measures are also explained and compared for each fuzzy set. The proposed similarity can be applied to all membership and non-membership points satisfying $0 \leq \mu(x) \leq 1\ and\ 0 \leq nu(x) \leq 1$. *Similarity measure has been designed for the membership and non-membership degree separately, and it constitutes a 2-dimensional component Together with two degrees, it is emphasized that the hesitation has relation with the similarity design. With the consideration of membership and non-membership degree, each similarity measure component is integrated as the ordered sets. Similarity measure integration with two measures provides several analysis outcomes; magnitude, 1-norm, and inner product with respect to* 45° *line. The obtained results can be included in the ordered set, and a lattice structure is proposed on the similarity measure. Additionally, the Cartesian product structure is organized for two similarity measures with the help of vector lattice structure. With the illustrative example, it is shown that the relevant result comparison is carried out with the existing result.*

Keywords: Intuitionistic Fuzzy Sets, Similarity Measure, Distance Measure, Membership Degree, Non-membership Degree, Lattice.

Mathematics Subject Classification : 68P10, 94A17

Journal of Economic Literature (JEL) Classification : D81, Y10

1. INTRODUCTION

On the fuzzy set analysis, most of works have been considered by standardized membership function structure, those are composed of membership and non-membership degree satisfying total unit information. Standard fuzzy membership grades in the fuzzy set were expressed with precisely (Zadeh, 1965). For attempting better description, the research has extended by including imprecision and uncertainty through approaching on intuitionistic and interval valued fuzzy subsets (Atanassov, 1986; AtanassovK. and GargovG., 1989; Atanassov, 2012; Karnik and Mendel, 2001; Mendel and John, 2002).

In order to discriminate the pattern, proper measure need to provide numeric value. Then, the similarity measure plays the degree shows numeric value between patterns similar degree. Statistical information such as mean, variance and correlation are utilized (Weiss, and Weiss, 2017). Furthermore, the heuristic approach is also considered to analyze the data characteristics (Abbass, Sarker and Newton, 2002). Heuristic approach methodology is considered by the design of entropy and similarity measure. Entropy and similarity measure are the essential for studying of data information including fuzzy sets. The characterization and quantification of data fuzziness have important role by applying the management of uncertainty in the modeling and design of many systems. As it was explained, the entropy is a measure of the data uncertainty, and the results have been established by previous researchers (Pal and Pal, 1989; Liu, 1992; Bhandari and Pal, 1993; Ghosh, 1995; Kosko and Burgess, 1998). Zadeh proposed fuzzy entropy as a measure of fuzziness; Pal and Pal analyzed classical Shannon information entropy; Kosko considered the relationship between distance measure and fuzzy entropy; Liu proposed axiomatic definitions of entropy, distance measures, and similarity measures and discussed the relationships among these three concepts. Bhandari and Pal presented a measure of fuzzy information for distinguishing between fuzzy sets. Further, Ghosh used fuzzy entropy in neural networks (Ghosh, 1995). Data analysis has been based on the analysis of calculating uncertainty and certainty of data. Uncertainty calculation with respect to fact can be measured from designing the fuzzy entropy. The results are emphasized by the design of fuzzy entropy with explicitly y (Luca and Termini, 1972; Lee, Kim, Cheon and Kim, 2005). By the complementary characteristics, similarity measure represents the degree of similarity between data sets (Chen and Chen, 2003; Li, Olson and Zheng, 2007; Lee, Kim and Choi, 2006; Lee, Pedrycz and Sohn, 2009). Hence, the entropy and similarity measure are explained with the complementary information for each other. So the similarity measure (entropy) could be derived from entropy (similarity measure) in the previous literature (Liu, 1992; Lee et al., 2006).

The similarity measure provides the degree of similarity between two or more data sets, and it has the central role in decision making, pattern classification, etc., (Rébillé, 2005; Sugumaran, Sabareesh and Ramachandran, 2008; Kang and Jin, 2008; Shih and Kai, 2008; Hsieh and Chen, 1999). Thus, the similarity measures design on FSs, IFSs and PFSs have been carried out by numerous researchers (Burillo and Bustince, 1996; Li and Cheng, 2002; Li et al., 2007; Hung and Yang, 2006; Szmidt and Kacprzyk, 2001; Yager, 2014; Wei and Wei, 2018; Zhang, Hu, Feng, Liu and Li, 2019). From the conventional researches, similarity measure has been derived via fuzzy numbers for the FSs (Chen and Chen, 2003). However, derived similarity measures are restricted to triangular or trapezoidal membership functions (Chen and Chen, 2003). Similarity measures design with the distance measure can be generalized and are applicable to the general fuzzy membership functions including nonconvex fuzzy membership functions (Lee et al., 2009; Lin, 2008). For the similarity measure on IFSs, measure design shows the structure with the difference with membership and non-membership, $|\nu_A(x_i) - \nu_B(x_i)|$ and $|\mu_A(x_i) - \mu_B(x_i)|$, respectively. Additionally, hesitation $\pi(x) = 1 - \mu(x) - \nu(x)$ is also expressed as similarity measure on IFSs (Burillo and Bustince, 1996). Recently, PFSs similarity measure is shown with the $\mu_P^2(x)$ and $\nu_P^2(x)$ (Wei and Wei, 2018; Zhang et al., 2019).

As an extension of fuzzy sets, IFSs and vague sets were introduced by Atanassov, Gau and Buehrer, respectively (Ghosh, 1995; Kosko and Burgess, 1998; Luca and Termini, 1972; Lee et al., 2005). Bustince and Burillo pointed out that the conventional comparison analysis between IFSs and vague sets showed the same (Chen and Chen, 2003). Object description through IFSs make more realistic, practical and accurate. Hence, fuzzy entropy and similarity construction on IFSs are important to get more reliable results. From the definition of IFSs, we have proposed fuzzy entropy and similarity measure design on IFSs (Park, Hwang, Park, Wei and Lee, 2013). In which results, fuzzy entropy was considered using the hesitation structure with respect to the crisp data. And the similarity measure is also proposed with the summation of hesitation between comparable IFSs. For the IFSs membership value viewpoint, all existing membership degrees are considered for positive value of $\mu(x)$, $\nu(x)$, and $\pi(x)$. When the $\mu(x)$ and $\nu(x)$ are close to degree one, it is possible to illustrate negative value of $\pi(x)$. It is also satisfied the case of $\mu(x)+\nu(x)+\pi(x)=1$.

In this regard, we introduce the characteristics the similarity measure on FSs, IFSs and PFSs. Due to the importance of similarity measure, we propose similarity measure design for the membership and non-membership range $0 \leq \mu(x) \leq 1$ and $0 \leq \nu(x) \leq 1$. The obtained similarity measure can apply to degree satisfying PFSs and over, that is, $\mu_P^2(x)+\nu_P^2(x)<1$ and $\mu_P^2(x)+\nu_P^2(x)>1$ as well. Distance measure helps us to construct similarity measure with explicitly, and with the formation of $\mu(x)$ and $\nu(x)$. Unlike with the conventional PFSs similarity measure, the similarity measure not include $\mu_P^2(x)$ and $\nu_P^2(x)$ even the considered degrees satisfy PFSs. The proposed similarity is verified by proof of its similarity measure definition. Further discussion on the hesitation $\pi(x)=1-\mu(x)-\nu(x)$ is delivered. With the reference of fuzzy line, hesitation value is divided into positive and negative, and the positive hesitation satisfy there is no information on the degree. Whereas negative hesitation satisfy the overlap with membership and non-membership degrees.

From the definition of entropy and similarity measure on fuzzy sets, entropy and similarity measure could be organized via distance measure (Liu, 1992). Furthermore, a relation between distance and similarity measures illustrated the total information (Atanassov, 2012). Membership and non-membership degree satisfy the range $0<\mu(x)<1$ and $0<\nu(x)<1$, respectively. Compare with the conventional similarity measure structure, two dimension similarity components are designed based on the membership and non-membership degree. The existing similarity has been designed by the combination of membership and non-membership difference with scalar value, in this case fuzzy set is considered for $\mu(x)+\nu(x) \leq 1$ or $\mu^2(x)+\nu^2(x) \leq 1$. Then the similarity measure represents as two ways, one is based on membership degree and the other is non-membership degree. They are independent components, hence the integrated similarity measure needs to be ordered through two similarity measures structure. It can be considered with magnitude, 1-norm structure or inner product to represent the order. The ordered set with inferior and superior operation constitutes lattice structure, and it propose the similarity measure order in non-empty set. With the graphical representation of lattice diagram, component ordering has been also illustrated.

In this paper, we emphasize the similarity measures on fuzzy sets, and it is verified via proof and applied to numeric data sets. In the next section, preliminary result on fuzzy sets and similarity measure are introduced. Membership degree is explained through graphical illustration together with non-membership degree. Furthermore, similarity measure definitions on FSs, IFSs and PFSs are introduced and discussed with the existing results. Three fuzzy sets, FSs, IFSs and PFSs are also illustrated with graphically. In Section 3, existing similarity measure on FSs, IFSs and PFSs are introduced and analyzed. And the similarity measure is proposed by the consideration for all membership $0 < \mu(x) < 1$ and non-membership is also satisfying $0 < \nu(x) < 1$. Then the similarity measure constitutes two components of membership and non-membership degree, and the unified similarity measure is proposed to constitute ordered set with lattice structure. In Section 4, ordered set in the considered inner product is illustrated which has order characteristics with magnitude and phase with respect to the linear line. Magnitude and phase are designed with membership and non-membership similarity measure. Additionally, the unified similarity is illustrated with Hasse diagram which represent the similarity measure order. Finally, the conclusions are stated in Section 5. In this paper, fuzzy set notations are followed the relevant references (Zadeh, 1965; Atanassov, 1986; Liu, 1992).

2. PRELIMINARIES ON FUZZY SETS AND SIMILARITY MEASURE

In this preliminary, fundamental knowledge on fuzzy sets and similarity measure are introduced. Characteristics on standard fuzzy sets, intuitionistic fuzzy sets and Pythagorean fuzzy sets are explained with its membership degrees, and the existing similarity measure.

2.1. Fuzzy sets and intuitionistic fuzzy sets

Zadeh introduced the fuzzy set with its membership function $\mu_A(x)$ for universe of discourse $x \in X$ with $\mu_A(x)$ belong to the value inbetween $[0,1]$. Fuzzy set A in the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ defined as follows:

$$A = \{\langle x, \mu_A(x)\rangle \mid x \in X, \mu_A(x) \in [0,1]\}$$

where, $\mu_A(x)$ denotes a membership function of x in X. And, non-membership is expressed for the normalized set as $\nu_A(x) = 1 - \mu_A(x)$.

As the extension of FSs, Atanassov introduced IFSs which includes hesitation of information, which is data uncertainty. From the definition, it is noticed that the information is categorized in more detail (Atanassov, 1986; AtanassovK. and GargovG., 1989). Together with membership degree $\mu_I(x)$, non-membership degree $\nu_I(x)$ is expressed in the following definition, respectively.

Definition 1. (Atanassov, 1986) IFSs I for the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ is defined as follows:

$$I = \{\langle x, \mu_I(x), \nu_I(x)\rangle \mid x \in X, \mu_I(x), \nu_I(x) \in [0,1], 0 \le \mu_I(x) + \nu_I(x) \le$$

where, $\mu_I(x)$ and $\nu_I(x)$ denote a membership function and non-membership function of $x \in X$, respectively.

From the Definition 1, it is clear that membership degree of IFS I should be restricted in $(\mu_I(x), \nu_I(x))$. Degree of uncertainty can be defined by $1 - \mu_I(x) - \nu_I(x) = \pi_I(x)$ Furthermore, if $\mu_I(x) + \nu_I(x) = 1$, then IFSs I is considered as a standard fuzzy set. To evaluate the uncertainty or entropy on IFSs, hesitance information, membership and non-membership degree have to be considered. By considering hesitance, fuzzy set property is defined.

Definition 2. (Atanassov, 1986) For IFSs I in the universe of discourse, if $\mu_I(x) + \nu_I(x) = 1$ and $\mu_I(x) + \nu_I(x) = 0$, then I is considered as a fuzzy set and null set, respectively.

All degree points in the whole range represent all values belong to $0 \le \mu(x) \le 1$ and $0 \le \nu(x) \le 1$. Hence, all points are defined as FSs, IFSs and PFSs by their definition. We illustrated the relation of membership and non-membership value with figure in later. In the reference (Lee, et al, 2005)], null set was modelled by having no any other information about data themselves. It represents the coordination of $\mu_I(x) - \nu_I(x)$ plane. Even the hesitance is illustrated by the area under the fuzzy line. Inside of the area, it is clear to obtain the relation of $\mu_I(x) + \nu_I(x) + \text{hesitation} = 1$. By the graphical representation, it is clear that hesitance satisfies one as $\mu_I(x) + \nu_I(x) \to 0$, that is, hesitancy approaches to origin; whereas sets on the fuzzy line means that it has no hesitance. Under the fuzzy line, relations between membership degree and non-membership degree are defined by $0 < \mu_I(x) + \nu_I(x) < 1$.

2.2. Pythagorean fuzzy sets

PFSs is introduced as another non-standard fuzzy sets. Information representation with more flexible expression, and it is illustrated by IFSs and interval valued fuzzy sets (Atanassov, 1986; Atanassov. and Gargov G., 1989). Specifically, PFSs extends the application range of IFSs by the comparison result between intuitionistic fuzzy number and Pythagorean fuzzy number (Yager, 2014; Wei and Wei, 2018).

Definition 3. (Szmidt and Kacprzyk, 2001 A PFS P for the universe of discourse $X = \{x_1, x_2, \dots, x_n\}$ is defined as follows:

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle \,|x \in X, \mu_P(x), \nu_P(x) \in [0,1], \mu_P^2(x) + \nu_P^2(x) \le 1\}$$

where, $\mu_P(x)$ and $\nu_P(x)$ denote a membership function and non-membership function of $x \in X$, respectively.

Membership relations are summarized as follows:

- IFSs: $\mu_I(x) + \nu_I(x)\pi_I(x) = 1$
- PFSs: $\mu_P^2(x) + \nu_P^2(x) + \pi_P^2(x) = 1$

From Definition 2 and 3, hesitation degree of IFSs and PFSs are defined as follows (Peng and Selvachandran, 2019):

- IFSs: $\pi_I(x) = 1 - \mu_I(x) - \nu_I(x)$
- PFSs: $\pi_P(x) = \sqrt{1 - \mu_P^2(x) - \nu_P^2(x)}$

For the specific $x \in X$, assuming $\mu_I(x) = \mu_P(x)$ and $\nu_I(x) = \nu_P(x)$, then

$$\sqrt{1 - \mu_P^2(x) - \nu_P^2(x)} \ge 1 - \mu_I(x) - \nu_I(x)$$

is satisfied for range $0 \le \mu(x) \le 1$ and $0 \le \nu(x) \le 1$

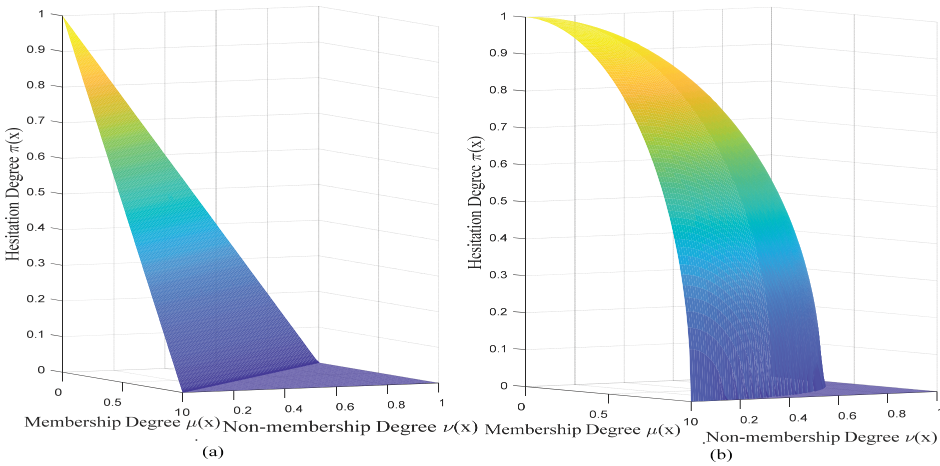


Figure 1. Hesitation $\pi_I(x)$ (a) and $\pi_P(x)$ (b).

From the Equation 1, it is notified that hesitation extend the range even with the same membership and non-membership degree, $\mu_P(x) = \mu_I(x)$ and $\nu_P(x) = \nu_I(x)$. It cause from the degree limitation, $0 \leq \mu(x) \leq 1$ and $0 \leq \nu(x) \leq 1$, then the square $\mu^2(x)$ and $\nu^2(x)$ are less than $\mu(x)$ and $\nu(x)$. From the simulation results on hesitation and hesitation difference are illustrated in Figure 1 and 2, respectively.

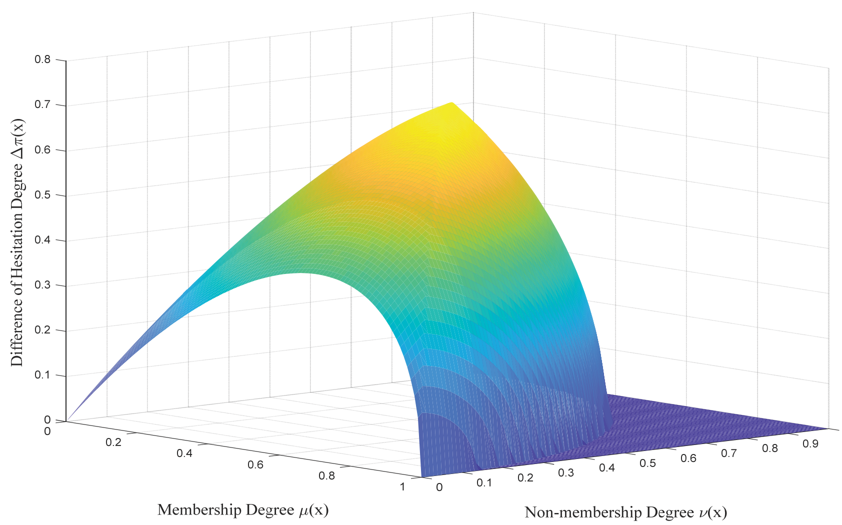


Figure 2. Difference between $\pi_P(x)$ and $\pi_I(x)$.

In the simulation, it is not considered for the case of $\mu_I(x) + \nu_I(x) + \pi_I(x) > 1$ and $\mu_P^2(x) + \nu_P^2(x) + \pi_P^2(x) > 1$. In the next subsection, similarity measure design procedure is shown with similarity definition and previous examples.

2.3. Similarity measure

Many researches on the similarity measure have been carried out by the numerous researchers, it was designed based on distance measure and fuzzy number r (Pal and Pal, 1989; Liu, 1992; Bhandari and Pal, 1993; Ghosh, 1995; Kosko and Burgess, 1998; Luca and Termini, 1972; Lee et al., 2005; Chen and Chen, 2003; Park, Lee, Song and Ahn, 2007; Lee et al., 2006). Similarity measure represent the similar degree between different information and data, and similarity measures design have been proposed based on the definition (Liu, 1992; Luca and Termini, 1972).

Definition 4. (Liu, 1992) A real function $s: F^2 \rightarrow R^+$ is called a similarity measure, ifs has the following properties:

(S1) $s(A, B) = s(B, A), A, B \in F(X)$,

(S2) $s(D, D^C) = 0, D \in P(x)$,

(S3) $s(C, C) = \max_{A,B \in F^S}(A, B), D \in F(X)$,

(S4) $A, B, C \in F(X)$, if $A \subset B \subset C$, then $s(A, B) \leq s(A, C)$, and $s(B, C) \geq s(A, C)$.

where $R^+ = [0, \infty)$, X is the universal set, $F(X)$ is the class of all fuzzy sets of X, $P(X)$ is the class of all crisp sets of X, and D^C is the complement of D.

The proposed similarity measure needs to be satisfied the Definition 1, and numerous similarity measures could be derived.

2.3.1 Similarity measure design on FSs with distance measure

In this subsection, the similarity measure for FSs is introduced with the distance measure. In order to illustrate with explicitly, distance measure is needed and the definition is introduced by Liu [40].

Definition 5. (Liu, 1992) A real function $d: F^2 \rightarrow R^+$ is called a distance measure on F, if d satisfies the following properties:

(D1) $d(A, B) = d(B, A)$, $A, B \in F(X)$,

(D2) $d(A, A) = 0, A \in F(X)$,

(D3) $d(D, D^C) = \max_{A,B \in F^d}(A, B)$, $D \in F(X)$,

(D4) $A, B, C \in F$, if $A \subset B \subset C$, then $d(A, B) \leq d(A, C)$, and $d(B, C) \leq d(A, C)$.

One of distance measure, Hamming distance is commonly used as distance measure between fuzzy sets A and B in the following equation:

$$d(A, B) = \frac{1}{n}\sum_{i=1}^{n}|\mu_A(x_i) - \mu_B(x_i)|$$

where $X = \{x_1, x_2, \dots, x_n\}$, $|k|$ was the absolute value of k. $\mu_A(x)$ is the membership function of $A \in F(X)$.

With the Definition 4, similarity measure is proposed. It can be represented as explicit structure, and the proposed similarity measures were illustrated in previous research (Park et al., 2007; Lee et al., 2006; Wang, Lee and Kim, 2009).

Theorem 1. (Wang et al., 2009) For any set $A, B \in F(X)$, if d satisfies Hamming distance measure, then

$$s(A,B) = d\big((A \cap B), [0]_x\big) + d\big((A \cup B), [1]_x\big)$$

is the similarity measure between set A and B.

Besides Theorem 1, numerous similarity measures are also possible to design. Other similarity measure shows other structures as in Theorem 2, and its proof is also found in previous result [3,4].

Theorem 2. (Park et al., 2007; Lee et al., 2009) For any set $A, B \in F(X)$, if d satisfies Hamming distance measure, then

$$s(A,B) = 1 - d(A, A \cap B) - d(B, A \cap B)$$

$$s(A,B) = 2 - d\big((A \cap B), [1]_x\big) - d\big((A \cup B), [0]_x\big)$$

where ∩ and ∪ are defined as minimum and maximum, respectively.

Similarity measures Equation 3 and 4 are illustrated by the combination of common and uncommon information between two fuzzy sets A and set B. The results on similarity measures are derived with the distance measure based computation of the degree of similarity. Liu has also proposed an axiomatic definition of the similarity measure for $\forall\, A, B \in F(X)$ and $\forall\, D$ in crisp set (Atanassov, 2012).

2.3.2 Similarity measure on IFSs

There are some researches on the analysis of IFSs entropy has been considered (Chen and Chen, 2003; Lin, 2008). They allow us to measure the degree of hesitation for the IFSs, and non-probabilistic type entropy measure with a geometric interpretation of IFSs. It was proposed an axiomatic definition of IFSs, which was considered by taking into account fuzzy set consideration.

Definition 6. (Chen and Chen, 2003) A real function $I\!:\!\mathrm{IFS(X)} \to R^+$ is called an entropy on IFS(X) if I has the following properties:

(IP1) $I(A) = 0$, if and only if A is a fuzzy set,

(IP2) $I(A) = cardinal(X) = N$, if and only if $\mu_A = \nu_A = 0$, $\forall x \in X$,

(IP3) $I(A) = I(A^C)$ for $A \in \mathrm{IFS(X)}$,

(IP4) if $A \prec B$, then $I(A) \geq I(B)$.

where $A \prec B$ denotes that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \leq \nu_B(x)$ for all $x \in X$, which means that IFS B has less hesitancy than IFS A. $\mu_A(x)$, $\nu_A(x)$, and $\pi_A(x)$ are the degree of membership, non-membership, and hesitation of x in A, that is expressed by $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$.

Szmidt and Kaprzyk also described the fuzzy entropy on IFSs by the ratio of intuitionistic fuzzy cardinalities (Szmidt and Kacprzyk, 2001). Their definition was interesting. Briefly, entropy of datum "F" in Figure 3, it is obvious IFS, was represented by

$$E = \frac{a}{b}$$

where a and b are distance from F to the nearer and farther point among “A'” and “B'”.

Actually, hesitation is contained in distance a and b. Hesitance is illustrated by the under fuzzy line; dotted line between “A'” and “B'” in Figure 3. Fuzzy entropy was proposed by De Luca and Termini and the axiomatic definition referred to Shannon’s probability entropy (Luca and Termini, 1972).
Next, another similarity measure between IFSs was introduced by Dengfeng and Chuntian, it has similar formulation with Definition 3 of the reference (Luca and Termini, 1972). Implicit similarity measure is defined in Definition 7, those are similar with Definition 5.

Definition 7. (Li and Cheng, 2002) A mapping S:$\mathrm{IFS(X)} \times \mathrm{IFS(X)} \rightarrow [0,1]$. IFSs(X) denotes the set of all IFSs in $X = \{x_1, x_2, \dots, x_n\}$. $s(A, B)$ is said to be the degree of similarity between $A \in \mathrm{IFS(X)}$ and $B \in \mathrm{IFS(X)}$,if $s(A, B)$ satisfies the properties of conditions:
(P1) $s(A, B) \in [0,1]$,
(P2) $s(A, B) = 1 \Leftrightarrow A = B$,
(P3) $s(A, B) = s(B, A)$,
(P4) $s(A, C) \leq s(A, B)$ and $s(A, C) \leq s(B, C)$ if $A \subset B \subset C$, $C \in \mathrm{IFS(X)}$,
(P5) $s(A, B) = 0 \Leftrightarrow A \in \Phi \; and \; B = \bar{A}, or \; \bar{B} \; and \; B \in \Phi$.

where Φ means that information on IFS is very clear, $\mu_I = 0$ and $\nu_I = 1$. $\bar{A}$ detnotes the IFS complement of A.

With the definition of fuzzy entropy and similarity measure, entropy and similarity measure have been designed (Pal and Pal, 1989; Xuecheng, 1992; Bhandari and Pal, 1993). Its effectiveness has been verified with the examples in the previous results. Comparison between IFSs similarity measure was done by Lin, Olson, and Qin (Lin, 2008). They have compared conventional similarity measures. Furthermore, other fuzzy entropy for IFSs has been also defined by Hung and Yang (Hung and Yang, 2006). In their definition, IP2 and IP4 are different from those of their properties. Difference of two definitions has their own characteristics.

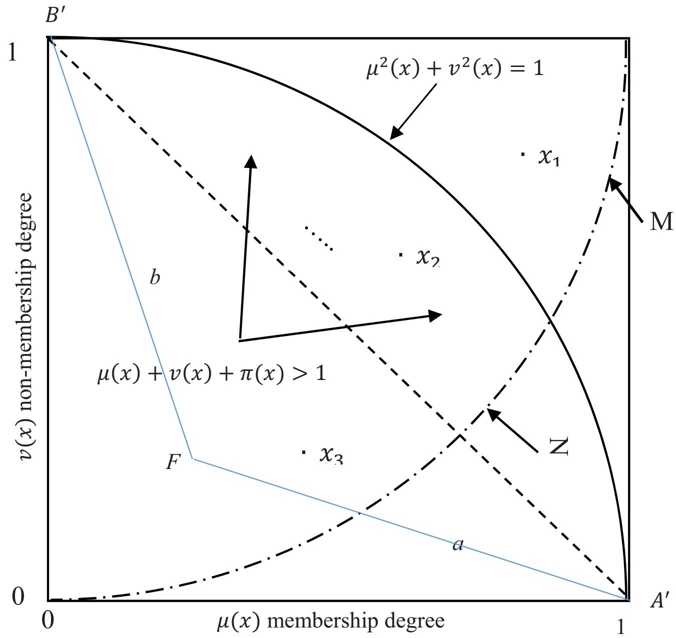


Figure. 3 Graphical representation of membership function and non-membership function.

Similarity measures for IFSs contained counter-intuitive cases (Lee et al., 2009). Especially, condition P2 is very strict to overcome, because similarity measure based on the difference calculation between two IFSs. For example, Lin et al showed similarity measure as follows (Li et al., 2007):

$$S_H(A,B) = 1 - \frac{\sum_{\{i=1\}}^{n}(|v_A(x_i) - v_B(x_i)| + |\mu_A(x_i) - \mu_B(x_i)|)}{2n}.$$

Other conventional similarity measure showed almost same results. Here novel similarity measure between IFSs is considered as (Park et al., 2013).

Theorem 3. (Park et al., 2013) Following equation satisfies a similarity measure on IFS(X).

$$S_L(A,B) = 1 - D_L(A,B)$$

where $D_L(A,B)$ is expressed by the hesitancy distance between two IFSs, that is,

$$D_L(A,B) = \frac{1}{n}\sum_{i=1}^{n} d\big(\pi_A(x_i), \pi_B(x_i)\big).$$

Then, similarity measure has the following explicit formulation.

$$S_L(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} d\big(\pi_A(x_i), \pi_B(x_i)\big).$$

Proof is delivered in (Park et al., 2013). The theorem derived by the consideration of hesitation distance.

2.4. Lattice structure

In order to understand the lattice ***L***, its basic concept is illustrated in the reference (Grãtzer, 2006). Now, we consider the order of each component in set. And the related knowledge is found in the relevant research (Priestley, 2002). Let ***L*** be an ordered set. From the mathematical definition on lattice, it is defined as; for the order $A = \langle A, \leq \rangle$, binary operation of $\leq$ has the relation with *reflexive*, *antisymmetric*, and *transitive* characteristics. For any ordered subset $\{x_i, x_j\} \subset \boldsymbol{L}$, there exist inferior and superior limits; it is expressed as:

$$x_i \wedge x_j\text{:inferior limit of } \{x_i, x_j\}$$

$$x_i \vee x_j\text{: superior limit of } \{x_i, x_j\}.$$

Thus, a lattice $\boldsymbol{L}$ is defined as:

$$x_i \in \boldsymbol{L} \text{ and } x_j \in \mathbf{L}, \forall\, x_i \text{and } x_j$$

$$\exists!\; x_k = x_i \wedge x_j \text{ and } x_k \in \boldsymbol{L}$$

$$\exists!\; x_l = x_i \vee x_j \text{ and } x_l \in \boldsymbol{L}$$

Definition 8. (Grãtzer, 2006) An order $\boldsymbol{L}$ is a lattice iff $a \wedge b$ snd $a \vee b$ always exist, for $\forall\, a, b \in \boldsymbol{L}$.

In lattice, binary operation means that it satisfies idempotent, commutative, associative and absorption identity (Grãtzer, 2007). Then for the following definition is also summarized from the derivation of lattice diagram (Grãtzer, 2007). Brief description of absorption identity follows as; $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ for $\forall\, a, b \in \boldsymbol{L}$.

Definition 9. (Grãtzer, 2007) An algebra $\boldsymbol{L} = \langle L, \wedge, \vee \rangle$ is a lattice iff L is nonempty set with ∧ and ∨ are binary operations on L and both ∧ and ∨ are idempotent, commutative, associative, and two absorption identities for each operation.

From the Definition 8 and 9, we illustrate lattice diagram. It is also denoted as *Hasse* diagram. With the ordered set $L_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, *Hasse* diagram is considered in Fig.4. Thus L_1 is at least a reflexive transitive partially ordered set, it is also called as *poset*. Using the notation $x_i \prec x_j$ if x_i precedes x_j; it is expressed for the ordered set L_1 in Figure 4.

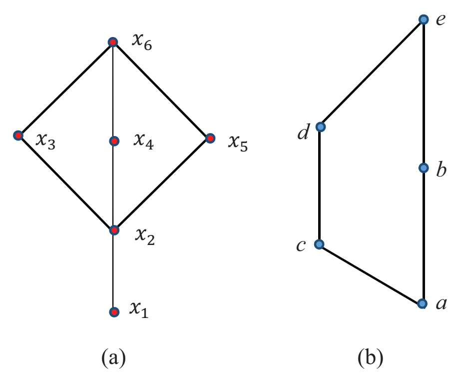


Figure 4. *Hasse* diagram with ordered set $\boldsymbol{L}_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ (a) and $\boldsymbol{L}_2 = \{a, b, c, d, e\}$ (b).

Then three possible relation satisfy with the help of preference $\prec$: $x_1 \prec x_2 \prec x_3 \prec x_6$, $x_1 \prec x_2 \prec x_4 \prec x_6$, and $x_1 \prec x_2 \prec x_5 \prec x_6$ in Figure 4(a). Another lattice $\boldsymbol{L}_2 = \{a, b, c, d, e\}$ is shown in Figure 4(b) with:

$$a \prec b \prec c \text{ and } a \prec c \prec d \prec e$$

is satisfied. Furthermore, the relation *b* and *c* is not comparable, *b* and *d* as well.

Then the product ordering $\boldsymbol{L}_1 \times \boldsymbol{L}_2$ can be illustrated:

$$(x_1, a) \prec (x_1, c) \prec (x_1, d) \prec (x\_1, e)$$

$$(x_1, a) \prec (x_2, a) \prec (x_3, a) \prec (x_6, a) \text{ and so on.}$$

When the poset include a greatest lower bound (glb) and least upper bound (lub) for each nonempty subset, it is a complete lattice.

3. SIMILARITY DESIGN ON FUZZY SETS

Similarity measure on Pythagorean fuzzy set is proposed by the consideration of previous similarity measure of IFSs and standard fuzzy set. First, the importance of the similarity design for the Pythagorean fuzzy sets is introduced and the measure design is delivered.

3.1. Analysis on membership and non-membership degree of FSs, IFSs, and PFSs

From the fuzzy set classification, we assume the membership and non-membership degree with $0 \leq \mu(x) \leq 1$ and $0 \leq \nu(x) \leq 1$, respectively. Then, several ranges are categorized:

$$\mu(x) + \nu(x) < 1, \text{positive hesitation}: \pi(x) > 0,$$

$$\mu(x) + \nu(x) = 1, \text{no hesitation}: \pi(x) = 0,$$

$$\mu(x) + \nu(x) > 1, \text{negative hesitation}: \pi(x) < 0.$$

Above three categories can be divided from the hesitation viewpoints. Specifically, negative hesitation range includes PFSs range as below;

$$\mu(x) + \nu(x) > 1, \text{and bound with } \mu(x) \leq 1 \text{ and } \nu(x) \leq 1.$$

Classification with the values of $\mu(x)$ and $\nu(x)$ are shown in Figure 3. Fuzzy set is considered as the standard IFSs, so the similarity measure has been designed with the Theorem 1 and 2 together with the references (Park et al., 2013). PFSs satisfying $\mu_P^2(x) + \nu_P^2(x) = 1$ has been proposed by Wei and Wei, Zhang *et al*. (Wei and Wei, 2018; Zhang et al., 2019). The difference of similarity with FSs, IFSs and PFSs is that all of the PFSs similarity measures include the combination of $\mu_P^2(x)$ and $\nu_P^2(x)$. And, it is still difficult to design the similarity measure for the fuzzy set having hesitation.

For the negative hesitation area also includes over PFSs range; greater than $\mu_P^2(x) + \nu_P^2(x) = 1$, it is noticed that the similarity measure design can be extended for the whole membership and non-membership range satisfying $\mu(x) + \nu(x) > 1$, $\mu(x) \leq 1$ and $\nu(x) \leq 1$. Specifically, it is interesting to investigate the similarity for the fuzzy set in the area over the beyond $\mu_P^2(x) + \nu_P^2(x) > 1$.

Additionally, there are two regions inside of Pythagorean membership degree. For the points inbetween $\mu(x) + \nu(x) = 1$ and $\mu_P^2(x) + \nu_P^2(x) = 1$, one of the point x_2 located in Figure 3. Under dotted line (Fuzzy line) represents positive hesitation; $\mu(x) + \nu(x) < 1$ then, $\pi(x) = 1 - \mu(x) - \nu(x)$ is positive. For the area in upper fuzzy line, hesitation satisfies negative as $\pi(x) = 1 - \mu(x) - \nu(x)$ for $\mu(x) + \nu(x) > 1$. Three points x_1, x_2 and x_3 are illustrated in membership and non-membership degree in Figure 3.

Except the fuzzy line $\mu(x) + \nu(x) = 1$, all degree points have hesitation $\pi(x)$. Which is expressed by:

$$\pi(x) = 1 - \frac{1}{n}\sum_{i=1}^{n}[\mu(x_i) + \nu(x_i)]$$

Hence, the mentioned hesitation has the value as positive or negative, then it plays an important role in similarity measure design. Positive and negative value act different meaning from the membership and non-membership degree viewpoint, positive value indicate there is no overlap between two degrees; membership and non-membership.

3.1. Similarity measure comparison for FSs, IFSs, and PFSs

Fuzzy sets and similarity measure definition and the conventional similarity measures are illustrated in the previous section. Recently, similarity measure on PFSs are appeared in the research (Yager, 2014; Wei and Wei, 2018; Zhang et al., 2019).

Characteristics of each similarity measure is summarized as follows:

- FSs similarity measure: hesitation $\pi(x) = 0$ is considered. Then the similarity measure constitutes with only $\mu(x)$ and $\nu(x)$ (Luca and Termini, 1972).
- IFSs similarity measure: in the conventional results are expressed with $\pi(x)$ (Yager, 2014).
- PFSs similarity measure: structure constitutes the difference between two PFSs with $\mu_P^2(x)$ and $\nu_P^2(x)$ (Wei and Wei, 2018; Zhang et al., 2019). It describes the minimal value of difference.

In the design of similarity measure, two approaches are considered. First, hesitation area minimization and shape difference are considered. FSs and IFSs similarity measure and entropy are designed by the consideration with $\mu(x)$ and $\nu(x)$. Even $\mu_P^2(x)$ and $\nu_P^2(x)$ are considered for the similarity measure on PFSs, it is also based on $\mu(x)$ and $\nu(x)$.

To design the explicit structure, distance measure is needed to illustrate the similarity between comparable data sets. Following distances are used in common; for $A = \{a_i|\langle a_1, a_2, .., a_i\rangle, i = 1, \dots, n\}$ and $B = \{b_i|\langle b_1, b_2, .., b_i\rangle, i = 1, \dots, n\}$,

$$d(A,B) = \frac{1}{n}\textstyle\sum_{i=1}^{n}|a_i - b_i|$$

$$d(A,B) = \sqrt{\frac{1}{n}\textstyle\sum_{i=1}^{n}(a_i - b_i)^2}.$$

Among the IFSs, there are different hesitation degree and need to be considerate for the similarity measure design. For example, two degree points *M* and *N* in Figure 3 show different hesitation. Point *M* include high membership and non-membership degree, furthermore high hesitation as well. Whereas Point *N* shows small hesitation, together with rather small membership and non-membership. In IFSs, from the fuzzy set properties, complement characteristic is not consistent, it means; $\mu_I^c(x)$ is not the same with $\nu_I(x)$. In FSs, $\mu^c(x) = \nu(x)$ and $\nu^c(x) = \mu(x)$, where superscript *C* denotes complement of membership degree and non-membership degree. FSs similarity measure design could be designed with single variable, $\mu(x)$ or $\nu(x)$ due to its complementary characteristic. However, two variable need to be considered by way of $\mu_I(x)$ and $\nu_I(x)$ for IFSs similarity measure design because of their independent characteristic, $\mu_I^c(x) \neq \nu(x)$ and vice versa.

Hence, similarity design and analysis on all degrees in Figure 3 are necessary by the consideration of $\mu_I(x)$ and $\nu_I(x)$. Now the similarity measure will be organized for the following structures:

$$s_1(A,B) = f\big(\mu_A(x), \mu_B(x)\big) \text{ and } s_2(A,B) = g\big(\nu_A(x), \nu_B(x)\big).$$

Next, explicit structure and analysis will be delivered.

3.1. Similarity measure with lattice

In this subsection, we propose the similarity measure design on similarity plane satisfying $0 \le \mu(x) \le 1$ and $0 \le \nu(x) \le 1$. Similarity measure design with only membership degree $\mu(x)$ is formulated as $s_1(A,B)$, whereas similarity measure design with only non-membership degree $\nu(x)$ is formulated as $s_2(A,B)$. With normalized similarity measure, it is applied to all degree points satisfying $0 \le s_1(A,B) \le 1$ and $0 \le s_2(A,B) \le 1$ and its formulation is applied to all degree point in Figure 3. By the similarity measure Definition 5, we consider two approach as usual; minimize hesitation $\pi(x)$ and, consider similar shape. Based on the conventional similarity with membership and non-membership degree, following similarity measure proposed.

Theorem 4. Following equation satisfies a similarity measure for all degree points in Figure 3**.**

$$s_1(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} |\mu_A(x) - \mu_B(x)|\,, \forall\, x_i \in X$$

where, $|\cdot|$ denotes absolute value, and $\mu_A(\cdot)$ and $\mu_B(\cdot)$ are membership degree of fuzzy set *A* and *B*. Membership degree replace with non-membership degree $\nu_A(\cdot)$ and $\nu_B(\cdot)$ makes non-membership similarity measure.

Proof. Similarity measure proof is clear with the Definition 5 and 7. From the Definition 5, (S1) is clear from the definition, $s(A,B) = s(B,A)$. And from the complement characteristic, crisp data satisfy $\mu_D(\cdot)$=1 and $\mu_{D^C}(\cdot) = 0$ or vise versa. Then, $\frac{1}{n}\sum_{i=1}^{n} |\mu_D(x_i) - \mu_{D^C}(x_i)| = 1$ is satisfied. For (S3), it is clear $\mu_A(x_i) - \mu_A(x_i) = 0$ for all $x_i \in X$. Finally, for fuzzy sets $A, B, C \in F(X)$, if $A \subset B \subset C$, then $s(A,B) \ge s(A,C)$ and $s(B,C) \ge s(A,C)$.
Similarly, similarity measure design on non-membership degree can be obtained by replace with $\nu_A(\cdot)$ and $\nu_B(\cdot)$.

Then from the value of similarity measure, point "*b*" in Figure 5 indicates high similarity on membership degree. Whereas "*c*" illustrates high similar on non-membership compare to the membership degree. High similar values are preferred with respect to membership and non-membership such as "*a*" and "*d*" points.
Then, similarity measures are become the component of two coordination, s_1 and s_2. Briefly, region *U* indicates high non-membership similarity, and high similar membership in region *D*. By the consideration of two dimension space:

$$s(A,B) = \big\{\langle\, s_1(A,B), s_2(A,B)\rangle \big| 0 \le\, s_{1(A,B)}, s_{2(A,B)} \le\, 1\big\}.$$

Each point, “*a*” to “*d*” is expressed by polar coordination as $s(A,B) = \sqrt{(s_1^2 + s_2^2)}e^{j\theta}$, where $\theta = tan^{-1}(\frac{s_2}{s_1}) = 45°$. Magnitudes in “*c*” to “*d*” are the same, but different membership and non-membership similarity illustrate respectively. And the points “*a*” and “*b*” has same membership similarity, but much difference in non-membership similarity. It is clear from the membership characteristic in Figure 5.

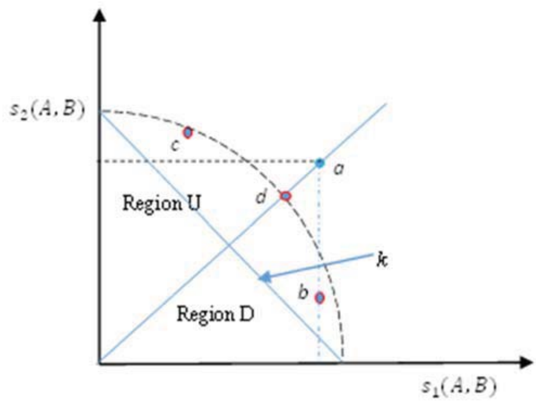


Figure 5. Membership and non-membership similarity measure.

From the knowledge of lattice structure in Subsection 2.4, two ordered set on similarity measure can be proposed:

$$\boldsymbol{L}_1 = \{s_1\} \text{ and } \boldsymbol{L}_2 = \{s_2\}$$

where s_1 and s_2 are similarity on membership degree and non-membership degree, respectively.

Ordering $\boldsymbol{L}_1 \times \boldsymbol{L}_2 = \{s_1, s_2\}$ can be considered with the consideration of measure on $\boldsymbol{L}_1 \times \boldsymbol{L}_2$ such as k_g and k_h:

$$k_g = |s_1| + |s_2| \text{and } k_h = \sqrt{s_1^2 + s_2^2}.$$

Then, we summarize the non-empty set *L* to institute lattice ***L***. For example, ordered set $\{k_g\}$ and $\{k_h\}$ have the components as: $L_{kp} = \{k_1, k_2, \ldots, k_n\}, \text{for } p = g, h \text{ all real values } n = 1,2, \ldots, \infty.$

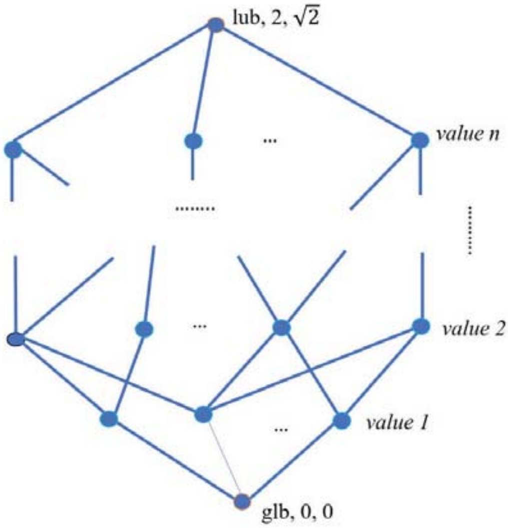


Figure 6. Lattice L_{kp} with ordered set $\{k_p\}$, $k_p = |s_1| + |s_2|$ and $k_p = \sqrt{s_1^2 + s_2^2}$.

Greatest lower bound (glb) and least upper bound (lub) of L_{kp} is summarize as follow: $\text{glb}(L_{kg}) = 0$, $\text{lub}(L_{kg}) = 2$ and $\text{glb}(L_{kh}) = 0$, $\text{lub}(L_{kh}) = \sqrt{2}$ are satisfied for $0 \le s_1 \le 1$ and $0 \le s_2 \le 1$. Then, $L_{kg} = \{0, \dots, 2\}$ and $L_{kh} = \{0, \dots, \sqrt{2}\}$ are expressed explicitly, and the lattice is illustrated in Figure 6.

Now we define inner product for each point with respect to the line which is satisfying $s_1 = s_2$. For the specific point in Figure 5, point $s = (s_1, s_2)$ is expressed as $s = \sqrt{s_1^2 + s_2^2} e^{j\theta}$; s_1 denotes membership similarity and s_2 is non-membership similarity; θ is the $tan^{-1}(\frac{s_2}{s_1})$.

Definition 10. Inner product between similarity measure and the proportional line is satisfied by the consideration of normalization of inner product:

$$s \cdot s_{REF} = |s||s_{REF}|\cos(\phi)$$

where $s = \sqrt{s_1^2 + s_2^2} e^{jtan^{-1}\left(\frac{s_2}{s_1}\right)}$, $S_{REF} = \frac{1}{\sqrt{2}} e^{j45^\circ}$, $\phi = |45^\circ - tan^{-1}(\frac{s_1}{s_2})|$ for $0 \le s_1 \le 1$ and $0 \le s_2 \le 1$.

We let the non-empty set L_{ke} such $\text{glb}(L_{ke}) = 0$, $\text{lub}(L_{ke}) = 1$, $L_{ke} = \{0, \dots, 1\}$. Inner product in Definition 10 provides its order with lattice, and it is illustrated in Figure 7.

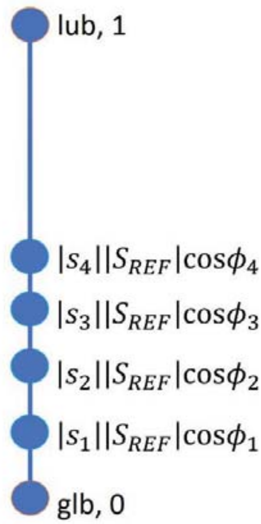
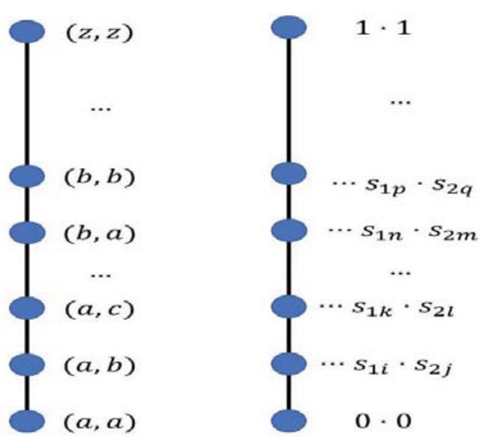


Figure 7. Lattice $\boldsymbol{L}_{ke}$ with order set $\{k_e\}$, $k_e = |s||s_{REF}|\cos(\phi)$

3.2. Relation analysis on two similarity measures

From the knowledge of lattice structure in Subsection 2.4, two ordered set on similarity measure can be proposed: $\boldsymbol{L}_1 = \{s_1\}$ and $\boldsymbol{L}_2 = \{s_2\}$. Where s_1 and s_2 are the similarity on membership degree and non-membership degree, respectively. Product lattice $\boldsymbol{L}_1 \times \boldsymbol{L}_2 = \{s_1, s_2\}$ can be expressed by dictionary as in Figure 8(a). where $\boldsymbol{L}_1 = \boldsymbol{L}_2 = \{a, b, c, \dots, z\}$.



(a) (b)

Figure 8. Dictionary order on the membership and non-membership similarity measure.

where $i, k, n, p, \dots$ belong to the set of membership degree similarities, and $j, l, m, q, \dots$ are parts of non-membership degree similarities. With the inner product definition, one of inner product is defined as:

$$s_{1i} \cdot s_{2j} = |s||s_{REF}|\cos(\phi)$$

where $s = \sqrt{s_{1i}^2 + s_{2j}^2} e^{jtan^{-1}\left(\frac{s_{2j}}{s_{1i}}\right)}$, $s_{REF} = \frac{1}{\sqrt{2}} e^{j45°}$, $\phi = \left|45° - tan^{-1}\left(\frac{s_{2j}}{s_{1i}}\right)\right|$. In Figure 8(b), infinite number of same values are per each level.

In this research, similarity measures design has been proposed for FSs which has membership and non-membership degree. Considered FSs ranges can cover IFSs and PFSs, and it is expected to be generalized. Designed similarity measures depend on membership and non-membership degree even IFSs and PFSs; membership degree $\mu(x)$ and non-membership degree $\nu(x)$; similarity measure for each membership and non-membership are considered as independently and separately. Hence, the proposed measure constitute with the two dimensional vector, so lattice structure or lexicographic are considered to decide the order.

For two lattice $\boldsymbol{L}_1 = \langle L_1, \leq \rangle$ and $\boldsymbol{L}_2 = \langle L_2, \leq \rangle$, two orders are considered as $\boldsymbol{L}_1 = \{s_1\}$ and $\boldsymbol{L}_2 = \{s_2\}$. When we consider s_1 and s_2 as membership and non-membership degree, then $\boldsymbol{L}_1 \times \boldsymbol{L}_2$ also constitute set with infinite elements; $k_g = |s_1| + |s_2|$ and $k_h = \sqrt{s_1^2 + s_2^2}$. Even for the $s \cdot s_{REF} = |s||s_{REF}|\cos(\phi)$, it constitutes infinite element in Figure 5. Lattice structure consist as follows; $L_1 = \{0, \sqrt{\frac{s_{11}^2+s_{21}^2}{2}}, \sqrt{\frac{s_{12}^2+s_{22}^2}{2}}, \ldots, 1\}$ and $L_2 = \{\cos(45°), \ldots, \cos(\phi_2), \cos(\phi_1), \cos(0°)\}$. For the component order follows:

$$0 < \sqrt{\frac{s_{11}^2 + s_{21}^2}{2}} < \sqrt{\frac{s_{12}^2 + s_{22}^2}{2}} < \cdots < 1$$

$$\text{and} \cos(45°) < \cdots < \cos(\phi_2) < \cos(\phi_1) < \cos(0°)$$

for $s_{11} < s_{12}, s_{21} < s_{22}, \ldots$ and $\phi_1 < \phi_2 < \cdots$. Then the product ordering $\boldsymbol{L}_1 \times \boldsymbol{L}_2$ can be constructed.

4. ILLUSTRATIVE EXAMPLES

In this section, numerical example is illustrated for the verification of the proposed similarity measure. And the similarity measure components are analyzed.

4.1. Numerical Examples

Example 1. Case of pattern recognition is considered; with the formation $A_i = \{\langle x_i, \mu_A(x_i), \nu_A(x_i)\rangle | x_i \in X\}$ (Wei and Wei, 2018). Three unknown patterns are represented as

$$A_1 = \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\},$$

$$A_2 = \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.1)\},$$

$$A_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\},$$

and the unknown pattern $B = \{(x_1, 0.5, 0.3), (x_2, 0.6, 0.2), (x_3, 0.8, 0.1)\}$.

Similarity measure calculation results are illustrated in the reference as well (Zhang et al., 2019). From the result, Zhang et al. showed similarity ranking with Pythagorean similarity measure as $A_3 > A_1 > A_2$, whereas existing results showed different order $A_3 > A_2 > A_1$. Calculation results depend on similarity measure structure.

The calculation result with Equation in Theorem 4 is illustrated in Table 1. For similarity measure with non-membership degree $\nu(x)$ is also used; $s_2(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} |\nu_A(x_i) - \nu_B\,(x_i)|$, $\forall x_i \in X$. Membership degree of A_1, A_2, A_3, *B* are satisfied by the characteristic degree of IFSs satisfying $\mu_A(x) + \nu_A(x) < 1$. Hence, the similarity measure keeps the same pattern for the similar distribution even the degree satisfies $\mu_A(x) + \nu_A(x) > 1$.

Table 1: Similarity calculation with Theorem 4.

Comparable sets	$s_1(A,B)$	$s_2(A,B)$	$s \cdot s_{REF}$
(A_1, B)	0.733	0.733	0.733
(A_2, B)	0.733	0.767	0.750
(A_3, B)	0.833	0.767	0.800
	Membership similarity	Non-membership similarity	Inner product

From Table 1, we show the result is not consistent for each similarity measure. Summation result is the same with the existing result (Wei and Wei, 2018; Zhang et al., 2019). And the similarity measure on PFSs, consisting component include $\mu_P^2(x)$ and $\nu_P^2(x)$. Hence, the measure calculation result can be changed by the measure structure, and the comparison pair shows not much difference. In Table 1, inner product $s \cdot s_{REF}$ calculations are obtained by:

$$s \cdot s_{REF_1} = \sqrt{\frac{0.733^2+0.733^2}{2}} \cos\left(\left|45° - tan^{-1}\left(\frac{0.733}{0.733}\right)\right|\right) = 0.733,$$

$$s \cdot s_{REF_2} = \sqrt{\frac{0.733^2+0.767^2}{2}} \cos\left(\left|45° - tan^{-1}\left(\frac{0.767}{0.733}\right)\right|\right) = 0.750,$$

$$s \cdot s_{REF_3} = \sqrt{\frac{0.833^2+0.767^2}{2}} \cos\left(\left|45° - tan^{-1}\left(\frac{0.767}{0.833}\right)\right|\right) = 0.800.$$

For the pattern recognition problem is compared with the previous example (Boran and Akay, 2014).

Example 2. (Boran and Akay, 2014) There are three known patterns P_1, P_2 and P_3 with the class label of C_1, C_2, and C_3 respectively. And these patterns are over an universe of discourse $X = \{x_1, x_2, x_3, x_4\}$, and they are represented by the IFSs as following:

$$P_1 = \{\langle x_1, 0.5{,}0.2|x_1 \in X\rangle, \langle x_2, 0.5{,}0.2|x_2 \in X\rangle, \langle x_3, 0.4{,}0.2|x_3 \in X\rangle, \langle x_4, 0.5{,}0.3|x_4 \in X\rangle\},$$

$$P_2 = \{\langle x_1, 0.5{,}0.3|x_1 \in X\rangle, \langle x_2, 0.5{,}0.2|x_2 \in X\rangle, \langle x_3. 0.4{,}0.2|x_3 \in X\rangle, \langle x_4, 0.3{,}0.5|x_4 \in X\rangle\},$$

$$P_3 = \{\langle x_1, 0.5{,}0.2|x_1 \in X\rangle, \langle x_2, 0.5{,}0.2|x_2 \in X\rangle, \langle x_3. 0.4{,}0.2|x_3 \in X\rangle, \langle x_4, 0.5{,}0.3|x_4 \in X\rangle\}.$$

Unknown pattern *Q* is represented by the IFSs as follows:

$$Q = \{\langle x_1, 0.4{,}0.2|x_1 \in X\rangle, \langle x_2, 0.5{,}0.2|x_2 \in X\rangle, \langle x_3, 0.4{,}0.2|x_3 \in X\rangle, \langle x_4, 0.5{,}0.5|x_4 \in X\rangle\}$$

The target of this example is to demonstrate that the pattern Q belongs to which class. To do so, the similarity calculation between unknown pattern Q and different classes C_1, C_2, and C_3 will be illustrated. Then pattern Q is assigned to class C_{i^*} described by the following equation:

$$i^* = \arg\left(\max_{1\le i\le 3}\left(S(P_i, Q)\right)\right).$$

In the calculation results (Boran and Akay, 2014), it was pointed out the unreasonable similarity degree by the different similarity measurements. In (Cheng, Chen, and Lan, 2016), S. M. Chen et al. has supplement some similarity measurements after Baron and Akay (Chen, 2003; Boran and Akay, 2014). The summarized results are illustrated in Table 2 with the proposed similarity measure. To make it consistent, the existing similarity measures are denoted by the same notations as them in (Cheng, Chen, and Lan, 2016).

Table2: The comparison of similarity measures (counter intuitionistic cases are in bold type).($p = 1$ in $S_M, S_{LS1}, S_{LS2}, S_{LS3}$ and $p = 1, t = 2$ in S_{BA})

	1	2	3	4	5	6
A	$(0.3,0.3)$	$(0.3,0.4)$	$(1,0)$	$(0.5,0.5)$	$(0.4,0.2)$	$(0.4,0.2)$
B	$(0.4,0.4)$	$(0.4,0.3)$	$(0,0)$	$(0,0)$	$(0.5,0.3)$	$(0.5,0.2)$
S_{BA}	0.967	0.9	0.5	0.833	0.967	0.95
S_C	**1**	0.9	0.5	**1**	**1**	0.95
S_{HK}	**0.9**	**0.9**	**0.5**	**0.5**	0.9	0.95
S_{FZ}	0.95	0.9	0.5	0.85	**0.95**	**0.95**
S_{LZD}	**0.9**	**0.9**	0.3	0.5	0.9	0.93
S_{DC}	**1**	**0.9**	0.5	**1**	**1**	0.95
S_M	**0.9**	**0.9**	**0.5**	**0.5**	0.9	0.95
S_{LS1}	**0.9**	**0.9**	**0.5**	**0.5**	0.9	0.95
S_{LS2}	0.95	0.9	0.5	0.75	**0.95**	0.95
S_{LS3}	0.93	0.933	0.5	0.7	0.93	0.95
S_{HY1}	**0.9**	**0.9**	0	0.5	**0.9**	**0.9**
S_{HY2}	**0.85**	**0.85**	0	0.38	**0.85**	**0.85**
S_{HY3}	**0.82**	**0.82**	0	0.33	**0.82**	**0.82**
S_Y	**1**	0.96	**0**	**0**	0.9971	0.9965
$s \cdot s_{REF}$	**0.9**	**0.9**	**0.5**	0.5	0.9	0.95

It could be found that the similarity degree obtained by the proposed similarity measure are counter intuitionistic when the FSs are in the neighborhood of the reference $\mu(x) = \nu(x)$ for single element in the pattern.

In Equation 17, $S(P_i, Q)$ denotes the similarity degree between unknown pattern and the known pattern P_i, where $i = 1,2,3$. The proposed similarity measurement and some existing similarity measurement summarized by Boran and Akay (Boran and Akay, 2014) are illustrated in Table 3.

It is obviously that S_C and S_{DC} cannot discriminate the P_2 and P_3 with unknown pattern Q. The similarity measure S_{FZ}, S_{LS2}, S_{HY1}, S_{HY2}, and S_{HY3} obtained the same similarity degree of pattern P_1 and P_2 with the unknown pattern Q. In this situation, the inner product $s \cdot s_{REF}$ can make it distinguishable for the unknown pattern Q with pattern P_1, P_2, and P_3 in different classes. It implies that the proposed similarity measure performed well for multi-elements in the pattern even the elements are characterized by the IFSs near to the reference $\mu(x) = \nu(x)$.

Table3: Similarity measures between the known patterns and unknown pattern in Example 2 (indistinguishable cases are in bold type). ($\mathrm{p} = 1$ in $\mathrm{S_M}, \mathrm{S_{LS1}}, \mathrm{S_{LS2}}, \mathrm{S_{LS3}}$ and $\mathrm{p} = 1, \mathrm{t} = 2$ in $\mathrm{S_{BA}}$)).

	$S(P_1, Q)$	(P_2, Q)	(P_3, Q)		$S(P_1, Q)$	(P_2, Q)	(P_3, Q)
S_C	0.963	**0.978**	**0.975**	S_{LS2}	**0.963**	**0.963**	0.950
S_{HK}	0.963	0.975	0.925	S_{LS3}	0.963	0.958	0.942
S_{FZ}	**0.963**	**0.963**	0.950	S_{HY1}	**0.925**	**0.925**	0.900
S_{LZD}	0.921	0.913	0.900	S_{HY2}	**0.886**	**0.886**	0.849
S_{DC}	0.963	**0.975**	**0.975**	S_{HY3}	**0.860**	**0.860**	0.818
S_M	0.963	0.950	0.925	S_Y	0.9917	0.9918	0.977
S_{LS1}	0.963	0.950	0.925	S_{BA}	0.963	0.967	0.958
$s \cdot s_{REF}$	0.963	0.950	0.925				

Example 3. (Cheng, Chen, and Lan, 2016) In this example, three known pattern $\widetilde{P_1}$, $\widetilde{P_2}$, and $\widetilde{P_3}$ are characterized by IFSs in the universe of discourse $X = \{x_1, x_2, x_3\}$, and they are belonged to three different classes $\widetilde{C_1}$, $\widetilde{C_2}$, and $\widetilde{C_3}$, respectively. These three patterns are represented as following:

$$\widetilde{P_1} = \{\langle x_1, 0.500, 0.450 | x_1 \in X\rangle, \langle x_2, 0.450, 0.500 | x_2 \in X\rangle, \langle x_3, 0.500, 0.500 | x_3 \in X\rangle\},$$
$$\widetilde{P_2} = \{\langle x_1, 0.425, 0.475 | x_1 \in X\rangle, \langle x_2, 0.450, 0.550 | x_2 \in X\rangle, \langle x_3, 0.450, 0.475 | x_3 \in X\rangle\},$$
$$\widetilde{P_3} = \{\langle x_1, 0.375, 0.525 | x_1 \in X\rangle, \langle x_2, 0.400, 0.600 | x_2 \in X\rangle, \langle x_3, 0.400, 0.525 | x_3 \in X\rangle\}.$$

The target of this example is to determine the unknown pattern $\tilde{Q}$ which is also represented by IFSs in the universe of discourse $X = \{x_1, x_2, x_3\}$ belonging to one of the classes $\widetilde{C_1}$, $\widetilde{C_2}$, and $\widetilde{C_3}$. The unknown pattern $\tilde{Q}$ is expressed as following:

$$\tilde{Q} = \{\langle x_1, 0.400, 0.500 | x_1 \in X\rangle, \langle x_2, 0.425, 0.075 | x_2 \in X\rangle, \langle x_3, 0.425, 0.500 | x_3 \in X\rangle\}.$$

To achieve the target, assignment function $s \cdot s_{REF}$ in Definition 10 is employed after the calculation of similarity degree. The results are illustrated in Table 4.

Table4: A comparison of the classification results of proposed similarity measure $s \cdot s_{REF}$ with existing similarity measures in Example 3 ($p = 1$ in S_{DC}, S_M, S_{LS1}, S_{LS2}, and S_{LS3}; $p = 2$ in S_L; $\omega_1 = \omega_2 = \omega_3 = \frac{1}{3}$ in S_{LS3}; $p = 1$ and $t = 2$ in S_{BA}; $w_1 = w_2 = w_3$ in S_{LS3}, S_{ZY}, S_{CC}, and S_{CCL}).

Similarity measure	$S(\widetilde{P_1}, \tilde{Q})$	$S(\widetilde{P_2}, \tilde{Q})$	$S(\widetilde{P_3}, \tilde{Q})$	Classification result
S_{BA}	0.8958	0.9083	0.8917	$\widetilde{C_2}$
S_C	0.8958	0.9083	0.8917	$\widetilde{C_2}$
S_{CC}	0.8837	**0.9552**	**0.9552**	Cannot be determined
S_{CR}	0.8634	0.8648	0.8426	$\widetilde{C_2}$
S_{DC}	0.8658	0.9083	0.8917	$\widetilde{C_2}$
S_{FZ}	0.8917	0.9042	0.8917	$\widetilde{C_2}$
S_{HK}	0.8875	0.9000	0.8917	$\widetilde{C_2}$
S_{HY1}	0.8000	0.8250	0.8083	$\widetilde{C_2}$
S_{HY2}	0.7132	0.7460	0.7241	$\widetilde{C_2}$
S_{HY3}	0.6667	0.7021	0.6783	$\widetilde{C_2}$

S_{LZD}	0.8177	0.8047	0.7845	$\widetilde{C_1}$
S_{LS1}	0.8875	0.9000	0.8917	$\widetilde{C_2}$
S_{LS2}	0.8958	0.9083	0.8917	$\widetilde{C_2}$
S_{LS3}	0.8958	0.9083	0.9000	$\widetilde{C_2}$
S_L	0.7362	0.7175	0.7040	$\widetilde{C_1}$
S_M	0.8875	0.9000	0.8917	$\widetilde{C_2}$
S_Y	0.9238	0.9184	0.8960	$\widetilde{C_1}$
S_{ZY}	**N/A**	**N/A**	**N/A**	Cannot be determined
S_{CCL}	0.9009	0.9222	0.9056	$\widetilde{C_2}$
$s \cdot s_{REF}$	0.8875	0.9000	0.8917	$\widetilde{C_2}$

Note:the counter intuitionistic cases are in bold type. '**N/A**' denotes that it cannot calculation because of 'the division by zero problem'.

From the results in Table 4, it is obviously that S_{LZD}, S_L, and S_Y are obtained the unreasonable result as the unknown pattern $\tilde{Q}$ assigned to class $\widetilde{C_1}$. S_{CC} and S_{ZY} cannot determine the pattern $\tilde{Q}$ belongs to which class in. The proposed similarity measure $s \cdot s_{REF}$ obtained the same result as majority similarity measures which is the unknown pattern $\tilde{Q}$ belonging to the class $\widetilde{C_2}$.

5. DISCUSSION AND CONCLUSION

Analysis on FSs, IFSs and PFSs has been done by reinforce on the membership and non-membership degrees. Relation with the degree of FSs, IFSs and PFSs are summarized through the Fig. 1, 2 and 4, and its characteristics are also analyzed. The existing similarity measures are also explained, the proposed similarity measure could be generalized over the IFSs and PFSs. It means that the proposed similarity could be applied to all membership degree $\mu(x)$ and non-membership degree $\nu(x)$ satisfying from zero to one.

In the research, we proposed two similarity measures on membership and non-membership degree separately. And the unified similarity measure is considered such as; 1-norm, magnitude and inner product. Then, by the continuous membership characteristics, lattice constitutes infinite points structure as in Fig. 5 and 6. By the comparison of the proposed similarity measure with existing ones, the proposed similarity measure performed well in the pattern recognition task. It avoids the counter intuitionistic cases in multi-elements pattern recognition.

6. REFERENCES

Abbass, H. A., Sarker, R. A. and Newton, C. S. 2002. Data mining: A heuristic approach, *Acoustics Speech & Signal Processing Newsletter* IEEE **27**(5): 310.

Atanassov, K. T. 1986. Intuitionistic fuzzy sets, *Fuzzy Sets & Systems* **20**(1): 87–96.

Atanassov, K. T. 2012. *On Intuitionistic Fuzzy Sets Theory*, Physica-Verlag HD.

Atanassov K. T and Gargov G. 1989. Interval valued intuitionistic fuzzy sets, *Fuzzy Sets and Systems* .

Bhandari, D. and Pal, N. R. 1993. Some new information measures for fuzzy sets, *Information Sciences* **67**(3): 209–228.

Boran, F. E. and Akay, D. 2014. A biparametric similarity measure on intuitionistic fuzzy sets with applications to pattern recognition, *Information Sciences* **255**: 45–57.

Burillo, P. and Bustince, H. 1996. Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets, *Fuzzy Sets and Systems* **78**(3): 305–316.

Chen, S.-J. and Chen, S.-M. 2003. Fuzzy risk analysis based on similarity measures of generalized fuzzy numbers, *IEEE Transactions on fuzzy system*s **11**(1): 45–56.

Cheng, S. H., Chen, S. M. and Lan, T. C. 2016. A novel similarity measure between intuitionistic fuzzy sets based on the centroid points of transformed fuzzy numbers with applications to pattern recognition, *Information Sciences* pp. 15–40.

Ghosh, A. 1995. Use of fuzziness measures in layered networks for object extraction: a generalization, *Fuzzy Sets & Systems* **72**(3): 331–348.

Grãtzer, G. 2007. Two problems that shaped a century of lattice theory, *Notices of the American Mathematical Society* **54**(6): 696–707.

Grãtzer, G. 2006. The Congruences of a Finite Lattice, A Proof-by-Picture Approach, *Inc., Boston, MA*.

Hsieh, C. H. and Chen, S. H. 1999. Similarity of generalized fuzzy numbers with graded mean integration representation, *Proc. 8th Int. Fuzzy Systems Association World Congr.,* **2**, 551-555

Hung, W.-L. and Yang, M.-S. 2006. Fuzzy entropy on intuitionistic fuzzy sets, *International Journal of Intelligent Systems* **21**(4): 443–451.

Kang, W. S. and Jin, Y. C. 2008. Domain density description for multiclass pattern classification with reduced computational load, *Pattern Recognition* **41**(6): 1997–2009.

Karnik, N. N. and Mendel, J. M. 2001. Operations on type-2 fuzzy sets, *Fuzzy Sets & Systems* **122**(2): 327–348.

Kosko, B. and Burgess, J. C. 1998. Neural networks and fuzzy systems, *The Journal of the Acoustical Society of America* **103**(6): 3131.

Lee, S.-H., Kim, J.-M. and Choi, Y.-K. 2006. Similarity measure construction using fuzzy entropy and distance measure, *International Conference on Intelligent Computing*, Springer, pp. 952–958.

Lee, S.-H., Kim, Y.-T., Cheon, S.-P. and Kim, S. 2005. Reliable data selection with fuzzy entropy, *International Conference on Fuzzy Systems and Knowledge Discovery*, Springer, pp. 203–212.

Lee, S.-H., Pedrycz, W. and Sohn, G. 2009. Design of similarity and dissimilarity measures for fuzzy sets on the basis of distance measure., *International Journal of Fuzzy Systems* **11**(2).

Li, D. and Cheng, C. 2002. New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions, *Pattern Recognition Letters* **23**(1/3): 221–225.

Li, Y., Olson, D. L. and Zheng, Q. 2007. Similarity measures between intuitionistic fuzzy (vague) sets: A comparative analysis, *Pattern Recognition Letters* **28**(2): 278–285.

Lin, S.-K. 2008. Gibbs paradox and the concepts of information, symmetry, similarity and their relationship, *Entropy* **10**(1): 1–5.

Luca, A. D. and Termini, S. 1972. A definition of a nonprobabilistic entropy in the setting of fuzzy sets theory, *Information and Control* **20**(4): 301–312.

Mendel, J. M. and John, R. B. 2002. Type-2 fuzzy sets made simple, *IEEE Transactions on fuzzy systems* **10**(2): 117–127.

Pal, N. R. and Pal, S. K. 1989. Object-background segmentation using new definitions of entropy, *IEE Proceedings. Part E* **136**(4): P.284–295.

Park, D.-h., Lee, S. H., Song, E.-H. and Ahn, D. 2007. Similarity computation of fuzzy membership function pairs with similarity measure, *International Conference on Intelligent Computing*, Springer, pp. 485–492.

Park, J.-H., Hwang, J.-H., Park, W.-J., Wei, H. and Lee, S.-H. 2013. Similarity measure on intuitionistic fuzzy sets, *Journal of Central South University* **20**(8): 2233–2238.

Peng, X. and Selvachandran, G. 2019. Pythagorean fuzzy set: state of the art and future directions, *Artificial Intelligence Review* **52**(3): 1873–1927.

Priestley, H. A. 2002*. Ordered sets and complete lattices: a primer for computer science*, Algebraic and coalgebraic methods in the mathematics of program construction.

Rébillé, Y. 2005. Decision making over necessity measures through the choquet integral criterion, *Fuzzy Sets & Systems* **157**(23): 3025–3039.

Shih, F. Y. and Kai, Z. 2008. A distance-based separator representation for pattern classification, *Image & Vision Computing* **26**(5): 667–672.

Sugumaran, V., Sabareesh, G. R. and Ramachandran, K. I. 2008. Fault diagnostics of rollerbearing using kernel based neighborhood score multi-class support vector machine, *Expert Systems with Applications* **34**(4): 3090–3098.

Szmidt, E. and Kacprzyk, J. 2001. Entropy for intuitionistic fuzzy sets, *Fuzzy Sets & Systems* **118**(3) : 467-477.

Wang, H., Lee, S. and Kim, J. 2009. *Quantitative comparison of similarity measure and entropy for fuzzy sets*, School of Mechatronics, Changwon National University Sarim-dong, Changwon, Gyeongnam, Korea;School of Mechatronics, Changwon National University Sarim-dong, Changwon, Gyeongnam, Korea;School of Mechatronics, Changwon National University Sarim-dong, Changwo .

Wei, G. and Wei, Y. 2018. Similarity measures of pythagorean fuzzy sets based on the cosine function and their applications, *International Journal of Intelligent Systems* **33**(3): 634–652.

Weiss, N. and Weiss, C. A. *Introductory statistics*, pearson etext, global edition, 10/e.

Liu, X. 1992. Entropy, distance measure and similarity measure of fuzzy sets and their relations, *Fuzzy sets and systems* **52**(3): 305–318.

Yager, R. R. 2014. Pythagorean membership grades in multicriteria decision making, *IEEE Transactions on Fuzzy Systems* **22**(4): 958–965.

Zadeh, L. A. 1965. Fuzzy sets, Information & Control **8**(3): 338–353.

Zhang, Q., Hu, J., Feng, J., Liu, A. and Li, Y. 2019. New similarity measures of pythagorean fuzzy sets and their applications, *IEEE Access* PP(99): 1–1.