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Design of Score Function for Decision Problem: with the Intuitionistic Fuzzy Sets Hesitation

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ABSTRACT

Hesitation analysis on the intuitionistic fuzzy sets (IFSs) has been carried out via hidden information and; negative and positive hesitation. Due to its casting vote role in decision, importance has been emphasized. Hesitation degree analysis is delivered from the mathematical definition, and the score function is designed based on the information overlap. Additionally, relation between fuzzy sets (FSs) and IFSs are also illustrated through figures, and each characteristic is also analyzed. Difference of FSs and IFSs are clear from the hesitation viewpoints, together with their similarity measure derivation. The obtained score function is applied to the example; multi-criteria decision problem. The proposed hesitation analysis provides more flexible and general results than FSs based score functions. In order to design score function, similarity measures are also considered by the existing results and our proposal. Classification results are illustrated in the examples, specifically dealing in purchasing product and decision making.

Keywords: Intuitionistic Fuzzy Sets; similarity measure; hesitation; membership degree; non-membership degree.
Mathematics Subject Classification: 62B10, 94A17

Journal of Economic Literature (JEL) Classification : D81, Y10

1. INTRODUCTION

The existing decision making has been emphasized in big data and management over the industry and networked system (Xiao et al., 2022; Gao et al., 2021). For the multi-criteria decision problem, score function is needed to evaluate attributes with respect to specific criterion. To evaluate the score among attributes, preference concept is necessary in advance (Grãtzer, 2007), which is used in the economics and management. To be more realistic, attribute preference; support, descent, absence values are needed to discriminate.

Herewith, fuzzy sets (FSs) and intuitionistic fuzzy sets (IFSs) are fitful to represent each attribute three status (Hong and Choi, 2000 ; Liu and Wang, 2007). Score function to order attribute for the specific criterion, measure functions has been considered with similarity measure, entropy, and accuracy function (Bhandari and Pal, 1993; Boran and Akay, 2014; Lee et al., 2009; Luca and Termini, 1972; Chen and Tan, 1994; Hong and Choi, 2000). Hence, the effective score function design is necessary to the multi-criteria problem.

In this research, we emphasize to design score function on the IFSs which investigate the role of hesitation specifically. Decision making is decisive how hesitation acts to membership function $\mu_A(x)$ or non-membership function $v_A(x)$. That is the importance of hesitation, hence more complete information and knowledge is needed with respect to $\mu_A(x)$ and $v_A(x)$. First, we will investigate hesitation on FSs and IFSs. In the analysis, recent our research output is added; hesitation analysis to positive and negative viewpoints. Negative hesitation represents information overlap between two existing information with respect to the comparable information (Yang et al., 2023). Whereas, positive hesitation indicates non-overlap to the two existing information. With existing research, we organize score function by the consideration of negative hesitation.

The score functions are also illustrated, and analyse for their realization and disadvantage. To complete multi-criteria decision problem, knowledge of lexicographic order also pointed out. Because of ordinal number on the criteria, preference between criteria is necessary. And the preference also transformed to the utility function to make order over the criteria. The lexicographic order is completed as accordingly; criteria order with utility function and attribute order with score function (Grãtzer, 2007).

The advantages and disadvantage of the existing researches are well addressed in the reference in (Wang and Li, 2011). It was pointed out the score functions and their comparisons, and the cross entropy was proposed to recover the disadvantages of the existing researches (Chen and Tan, 1994; Hong and Choi, 2000; Li and Rao, 2001; Zhang et al., 2006). Wang and Li proposed cross entropy based on the Ye's fuzzy cross entropy of vague sets, and it was classified by the size of affirmative and disssetation (Wang and Li, 2011; Ye, 2007).

In this regard, we consider the concept of affirmation and dissentation with the membership and non-membership by IFSs. And the comparison of $\mu_A(x)$ and $v_A(x)$ indicate the possibility of hesitation the specific direction by the multiplication. From the qualitative viewpoint, optimistic attitute is close to $\mu_A(x)$. It is expressed by $(\mu_A(x) - v_A(x))$ times hesitation components - $\pi_A(x)$. As the higher membership, affirmative or dissentient are decided as the scoring function value. More details on the derivations are delivered in Section 3. In the research, three types of expectation on the decision are also explained; $\mu_A(x) > v_A(x)$; $\mu_A(x) < v_A(x)$; $\mu_A(x) = v_A(x)$. It is the same justification of Wang and Li (2011). However, the calculation is rather complex. The proposed result is simple to realize the score function.

This paper is organized as follows. In Section 2, the hesitation analysis has been carried out on IFSs. And the comparisons are also noted. Hesitation analysis is delivered by the data overlap and non-overlap; it is classified as negative and positive hesitation. In Section 3, score functions are introduced to the decision problem. First, existing similarity measures are introduced to use as the score function. And the modified score function is proposed to apply multi-criteria problem. In order to apply multi-criteria problem, lexicographical order is also needed to order the criteria as priority. In Section 4, multicriteria decision-making problems are illustrated. Decision results show clear discrimination. Finally, conclusions are included in Section 5.

2. PRELIMINARIES ON HESITATION OF FUZZY SETS

Hesitation analysis based on the FSs is introduced. Even it is defined from IFSs, difference analyses between two sets are more than important to design the similarity measure for each set. Graphical illustration shows the hesitation area, and it is also related with score function design.

2.1. Fuzzy sets and intuitionistic fuzzy sets

FSs and IFSs are proposed to describe the object or information from the heuristic point of view with membership functions; membership function $\mu_A(x)$ and non-membership function $v_A(x)$ on FS; hesitation $\pi_A(x)$ for the fuzzy set A has been defined over the universe of discourse $x \in X$. $\mu_A(x)$ and $v_A(x)$ which are belong to the value inbetween $[0, 1]$. From the specific relation in IFSs, $\pi_A(x) = 1 - \mu_A(x) - v_A(x)$ is proposed as the hesitation degree by Atanassov (Atanassov, 1986, 1999, 2012). However, conflict can happen from the definition.

Based on the range definition of $\mu_A(x)$ and $v_A(x)$, hesitation could be expressed as positive and negative values. And they also include different information for each. In this regard, we delivered recent research output (Yang et al., 2023).

It is clear that $\mu_A(x) + v_A(x) = 1$ means FSs A is considered as a standard even membership degree of IFSs A should be restricted in $(\mu_A(x), v_A(x))$ - no hesitation. To evaluate the uncertainty or entropy on IFSs, $\pi_A(x)$ have to be considered together.

Definition 1 (Atanassov, 1986). For IFS A in the universe of discourse $x \in X$, when A is considered as a fuzzy set and null set if $\mu_A(x) + v_A(x) = 1$ and $\mu_A(x) + v_A(x) = 0$, respectively.

From the IFSs definition, when $\mu_A(x) + v_A(x) + \pi_A(x) = 1$ is satisfied. By the consideration of range of $\mu_A(x)$ and $v_A(x)$, $\pi_A(x)$ could be expressed as positive and negative. Hence, a negative hesitation is defined;

Definition 2 (Yang et al., 2023). A negative hesitation fuzzy set (NHFS) for set A is defined as:

$$A = \{(x, \mu_A(x), v_A(x), \pi_A^n(x)) | x \in X\}$$

where $\mu_A(x) \in [0,1]$, $v_A(x) \in [0,1]$, and $\pi_A^n(x) \in [-1,0]$. $\mu_A(x) + v_A(x) + \pi_A^n(x) \leq 1$ and the positive hesitation degree $\pi_A^p(x) = 1 - (\mu_A(x) + v_A(x) + \pi_A^n(x))$ is satisfied.

To organize $\pi_A^n(x)$, we need to define the information overlap between comparable data by the Definition 2. For an arbitrary element x with the projection of T in X, the negative hesitation degree $\pi_A^n(x)$ in the set A whose projection is C_1 can be defined as

$$\pi_A^n(x) = \pi_{C_1}^n(T) = -\frac{O(T, C_1 \cap C_2)}{O(T,T)} = -\frac{O(T, E_{12})}{O(T,T)} \tag{1}$$

where $\mathcal{O}(\cdot)$ denotes the overlap area function, C_1 and C_2 are the projections of the binary information distribution. And E_{12} is the elementary information derived from C_1 and C_2. More detail can be found in the reference in (Yang et al., 2023).

2.2. Hesitation realization on IFSs

Consider the component in the graphical representation of four information types in Fig. 1 (Lee and Yang, 2024). In Fig. 1, C_1 and C_2 represent two known set or information in the universe of discourse X. It is clear that $\mu(x)$ and $\nu(x)$ for C_1 and C_2 can be delivered as accordingly (Lee and Yang, 2024); there is no hesitation; $\pi(x) = 0$.

With adding the comparable information T, it is possible to investigate how much hesitation $\pi(x)$ is included with $\mu(x)$ and $\nu(x)$ for C_1 and C_2. For the comparable information, F and G satisfying $C_1 \subset F \subset C_1 \cup T \subset$ X, $C_2 \subset G \subset C_2 \cup T \subset$ X defined by Atanasov, $\mu(x)$ and $\nu(x)$ of the set C_1, μ_{c_1} and ν_{C_1} are obtained with the help of membership and non-membership function design procedure. Here we assume that C_1, C_2 and T represent the information as area, and normalized.

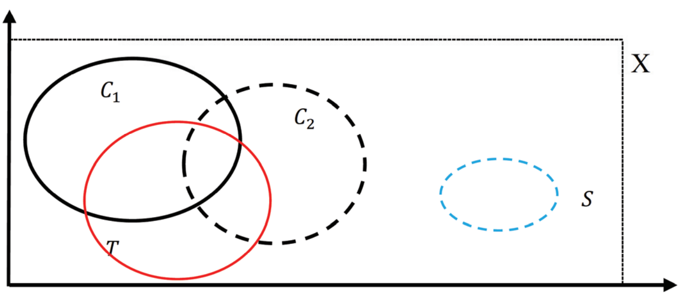

Fig. 1. Information Illustration (Lee and Yang, 2024)

From the membership and non-membership design procedure provides $\mu(x)$ and $\nu(x)$ of the set C_1;

$$\mu_{c_1} = \frac{T \cap C_1}{T} \tag{2}$$

$$\text{and } \nu_{C_1} = \frac{T \cap C_2}{T}. \tag{3}$$

Where the overlap area $\mathcal{O}(\cdot)$ to the numerator and denominator are deleted for the simplicity.

Then, the hesitation on set C_1 can be derived from the IFSs characteristic;

$$\pi_{c_1} = 1 - \mu_{c_1} - \nu_{C_1} = 1 - \left\{\frac{T \cap C_1}{T} + \frac{T \cap C_2}{T}\right\}$$

$$= 1 - \frac{T \cap (C_1 \cup C_2)}{T} - \frac{T \cap (C_1 \cap C_2)}{T}$$

Hence, it becomes

$$\pi_{c_1} = \frac{T \cap (C_1 \cup C_2)^c - T \cap (C_1 \cap C_2)}{T}. \tag{4}$$

From the negative and positive hesitation concept, each hesitation follows as follows;

$$\pi_{C_1}^n = -\frac{T\cap(C_1\cap C_2)}{T} \text{ and } \pi_{C_1}^p = \frac{T\cap(C_1\cup C_2)^c\cap T}{T}.$$

The result is generalized by the consideration of additional information – blue circle S in Fig. 1. There is no overlapped into C_1 and C_2, hence For the case, $\pi_{C_1}^n = \phi$ is empty because of $(C_1 \cap C_2) \cap T = \phi$, furthermore $\pi_{C_2}^n = \phi$ also satisfied.

Remark 1. For two comparable information F and G; $F \subset C_1 \cup T \subset$ X, $G \subset C_2 \cup T \subset$ X. There is no hesitation, it means $\mu_F(x) + \nu_F(x) = 1$ and $\mu_G(x) + \nu_G(x) = 1$. Hesitation constitutes of negative and positive; negative as overlap area; positive as the non-overlap area.

Remark 2. Comparable data can be extended to the multiple as the form of $T = \sum_{i=1}^n T_i$, where T_i can be overlapped or not.

3. SCORE FUNCTION DESIGN WITH HESITATION DEGREE

Score function design for the multi-criteria decision problem has been proposed. With the obtained property on the hesitation, score function is derived. The existing score function is also illustrated to compare its effective and drawback.

3.1. Score functions on IFSs

Score functions have been applied to decision problem (Chen and Tan, 1994; Hong and Choi, 2000; Gao et al., 2021; Liu and Wang, 2007; Wang and Li, 2010). To design score function with numeric value, it need to utilize the operator $\prec$ in advance; preference between two attributes. $A \prec B$ indicates that the score of A is less than that of B for A and B in IFSs. Existing score functions are illustrated and discuss on their structure as follows; the score function $S_C(A)$ by Chen and Tan (1994),

$$S_C(A) = \mu_A(x) - \nu_A(x) \tag{5}$$

where $S_C(A)$ exists over $[-1,1]$. By the comparison with $S_C(A)$ and $S_C(B)$, it has the same preference relation; $S_C(A) < S_C(B) \equiv A \prec B$. However, Eq. (5) faces difficulty when it has same $S_C(A) = S_C(B)$. Same difference between two membership function can effect too many different meaning; $\mu_A(x) = 0.7$ and $\nu_A(x) = 0.3$; $\mu_A(x) = 0.5$ and $\nu_A(x) = 0.1$ in IFSs. Even it has same score function, it includes difference preference or meaning. Additional accuracy functions are considered to avoid the conflict.

- A accuracy function $H(A)$ was proposed to overcome this difficulty, when $H(A) \in [0,1]$ is satisfied (Hong and Choi, 2000):

$$H(A) = \mu_A(x) + \nu_A(x) \tag{6}$$

The closer $H(A)$ to one, the more information have. Which is used to supplement decision problem.
A score function $S_L(A)$ was also proposed by (Liu and Wang, 2007):

$$S_L(A) = \mu_A(x) + \mu_A(x)\pi_A(x). \tag{7}$$

Which added by the hesitation emphasis on membership; rather optimistic viewpoints. But the abstaining dgree leads to conflict in zero membership degree ; $\mu_A(x) = 0$.
Cross entropy also considered to categorize a score function by the comparison of membership and non-membership value (Wang and Li, 2010):

$$S_W(A) = \begin{cases} \mu_A(x) - \nu_A(x) + E(x)\pi_A(x), & \mu_A(x) > \nu_A(x) \\ \mu_A(x) - \nu_A(x) - E(x)\pi_A(x), & \mu_A(x) < \nu_A(x) \\ 0, & \mu_A(x) = \nu_A(x) \end{cases} \tag{8}$$

where $E(x)$ denotes the cross-entropy. However, it has the problem in dealing with IFSs in particular cases, which had been illustrated in the conclusions of (Hong and Choi, 2000).
Gao et al. proposed a score function together with three degrees, $\mu_A(x)$, $\nu_A(x)$ and $\pi_A(x)$ (Gao et al., 2021):

$$S_G(A) = \frac{e^{\mu_A(x)-\nu_A(x)+\pi_A(x)\left(\mu_A(x)-\nu_A(x)\right)^3}}{1+\pi_A(x)} \tag{9}$$

However, it is not effective for the similar value of $\boldsymbol{\mu_A(x)}$ and $\boldsymbol{\nu_A(x)}$; $\boldsymbol{S_G}$ sometimes requires a precision up to $\mathbf{10^{-6}}$, it is not realistic.

3.2 Score function design with hesitation

Score function can be used to represent the ordinal number for each attribute under single criterion or multi-criteria. Hence the score function represent the closedness between criterion and attribute character. In this regard, similarity measure between attributes and criteria play a role of score function. Many types of similarity were reported on Ifs and IFSs; (Pal and Pal, 1989; Jung, Choi, Park and Lee, 2011).

- It satisfies similarity measure on FSs (Jung, Choi, Park and Lee, 2011). For any set $A, B \in F(X)$, $d(A,B) = \frac{1}{n}\sum_{i=1}^{n} |\mu_A(x_i) - \mu_B(x_i)|$ satisfies Hamming distance measure, then

$$s(A,B) = d\big((\mu_A(x) \cap \mu_B(x)), [0]_X\big) + d((\mu_A(x) \cup \mu_B(x)), [1]_X),\ x \in X \tag{10}$$

Where $\mu_A(x) \cap \mu_B(x)$ and $\mu_A(x) \cup \mu_B(x)$ are the min($\mu_A(x)$, $\mu_B(x)$) and max($\mu_A(x)$, $\mu_B(x)$), respectively. $[0]_X$ and $[1]_X$ denote the zero and one over the universe of discourse.

- Lin et al. showed similarity measure on IFSs as follows (Li et al., 2007):

$$S_H(A,B) = 1 - \frac{\sum_{\{i=1\}}^{n}(|\nu_A(x_i)-\nu_B(x_i)|+|\mu_A(x_i)-\mu_B(x_i)|)}{2n}. \tag{11}$$

- Park et al. provide following similarity measure on IFS(X);

$$S_L(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} d\big(\pi_A(x_i), \pi_B(x_i)\big). \quad (12)$$

Satisfying $S_L(A,B) = 1 - D_L(A,B)$, where $D_L(A,B) = \frac{1}{n}\sum_{i=1}^{n} d\big(\pi_A(x_i), \pi_B(x_i)\big)$ (Park et al., 2013).

From the illustrated similarity measure, we can propose more measure to satisfy the similarity measure definition (Liu, 1992; Chen and Chen, 2003; Li and Cheng, 2002).
Based on the hesitation information, we now proposed the score function with the following principles;

(P1) Scoring function is designed based on IFSs
(P2) Score function of an IFS set A is designed based on $\mu_A(x)$, $\nu_A(x)$ and $\pi_A(x)$
(P3) IFSs include the relation with total information; $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$
(P4) $\pi_A(x)$ affects the degree to $\mu_A(x)$ and $\nu_A(x)$

Theorem 1. Following function provides score on attributes to the specific criterion. It is considered based on (P1) to (P4).

$$S_{SH}(A) = (\mu_A(x) - \nu_A(x))e^{\pi_A(x)}. \quad (13)$$

Eq. (13) avoid typical case of Eq. (5) (Chen and Tan, 1994) ; $\mu_A(x) = 0.7$ and $\nu_A(x) = 0.2$; $\mu_A(x) = 0.6$ and $\nu_A(x) = 0.1$; they show same score. However, it can be discriminate with Eq. (13) ; $(0.7 - 0.2) \cdot 0.1 < (0.6 - 0.1) \cdot 0.2$ because of high hesitation and less descent. In this case, it is concluded by the following remark.

Remark 3. Hesitation affects to the score values, which depends on membership and non-membership together. High hesitation and less descent provides rather high score when $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$.

Furthermore, it is also possible to calculate in the case of no hesitation ; $\pi_A(x) = 0$.
It is also notified $S_{SH}(A) \in [-1, 1]$ for the negative value, $\mu_A(x) < \nu_A(x)$. It needs more numerical analyzing how much effect from $\mu_A(x)$ and $\nu_A(x)$ to score function value. For the membership and non-membership degree, $\mu_A(x) = 0.7$ and $\nu_A(x) = 0.2$; $\mu_A(x) = 0.6$ and $\nu_A(x) = 0.1$. Score function provides the difference between IFSs even $\mu_A(x) - \nu_A(x)$ is the same.

3.3 Score function application to the multi-criteria problem

In multi-criteria problem from the decision making, it need to data processing to IFSs formation. As it mentioned, support, descent, absence attitute with respect to the specific criterion become the foundation to design scoring function with IFSs.
Beside of ordinal numbering to attributes in specific criterion, preference ordering on criteria is also necessary to decide prior attribute under the multi-criteria. So, the data leicographical order can be considered. It has the lattice structure, hence brief introduction delivered in (Lee and Yang, 2004).

Let ***L*** be an ordered set, lattice is defined as; for the order $A = \langle A, \leq \rangle$, binary operation of $\leq$ has the relation with *reflexive*, *antisymmetric*, and *transitive* characteristics. For example, any ordered subset $\{x_i, x_j\} \subset \boldsymbol{L}$, there exist inferior and superior limits; it is expressed as:

$$x_i \wedge x_j\text{:inferior limit of } \{x_i, x_j\}$$
$$x_i \vee x_j\text{: superior limit of } \{x_i, x_j\}.$$

Definition 3. (Grãtzer, 2006) An order L is a lattice iff $a \wedge b$ snd $a \vee b$ always exist, for $\forall\, a, b \in \boldsymbol{L}$.

In lattice theory, binary operation satisfies idempotent, commutative, associative and absorption identity (Grãtzer, 2007); $a \vee (a \wedge b) = a$ and $a \wedge (a \vee b) = a$ for $\forall\, a, b \in L$.
From the Definition 3, we illustrate lattice diagram; *Hasse* diagram. With the ordered set $L_1 = \{c_1, c_2, \dots, c_n\}$ and $L_2 = \{a_1, a_2, \dots, a_m\}$, From Hasse diagram in Figure. 2, L_1 and L_2 are expressed ordered set simply. It is replaced the criterion preference $c_i \prec c_j$ if c_j precedes c_i; attributes order $a_i < a_j$ if a_j priors to a_i in L_2.

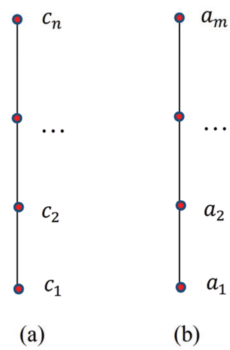

Figure 2. *Hasse* diagram with ordered set $L_1 = \{c_1, c_2, \dots, c_n\}$(a) and $L_2 = \{a_1, a_2, \dots, a_m\}$ (b)

To order attributes under the multi-criteria problem, we consider Cartesian product to two lattices; $L_1 \times L_2$. Then Cartesian product indicate as (c_i, a_j), for $i \in \{1, 2, \dots, n\}$ and $j \in \{1, 2, \dots, m\}$. As mentioned before criteria can be ordered with their preference, and attributes order can be obtained via score function we proposed.
Simply, we can it can be illustrated as the dictionary order for specific criterion $p \in \{1, 2, \dots, n\}$ as;

$$(c_p, a_i), (c_p, a_j), \dots, (c_p, a_k) \text{ for } i, j, k \in \{1, 2, \dots, m\}$$

If we say the criterion preference $p, q \in \{1, 2, \dots, n\}$ as $p \prec q$, then

$$(c_q, a_i), (c_q, a_j), \dots, (c_p, a_k) \text{ for } i, j, k \in \{1, 2, \dots, m\}$$

are followed. More clear ordinal ordering will be followed in the successive examples.

4. ILLUSTRATIVE EXAMPLES

In this section, score function is applied to the multi-criteria problem. Calculation results analysis and discussions are also followed.

4.1. Single criterion problem

Now we use the problem mentioned in (Wei and Wei, 2018). With the IFS structures;

$$A_1 = \{(x_1, 1.0, 0.0), (x_2, 0.8, 0.0), (x_3, 0.7, 0.1)\},$$
$$A_2 = \{(x_1, 0.8, 0.1), (x_2, 1.0, 0.0), (x_3, 0.9, 0.1)\},$$
$$A_3 = \{(x_1, 0.6, 0.2), (x_2, 0.8, 0.0), (x_3, 1.0, 0.0)\}.$$

From the results of Eq. (13), it shows via $S_{SH}(A) = (\mu_A(x) - \nu_A(x))e^{\pi_A(x)}$;

$$S_{SH}(A_1) = \frac{1}{3}\sum_{i=1}^{3} \left(\mu_{A_{1i}}(x) - \nu_{A_{1i}}(x)\right) e^{\pi_{A_{1i}}(x)}$$
$$S_{SH}(A_2) = \frac{1}{3}\sum_{i=1}^{3} \left(\mu_{A_{2i}}(x) - \nu_{A_{2i}}(x)\right) e^{\pi_{A_{2i}}(x)}$$
$$S_{SH}(A_3) = \frac{1}{3}\sum_{i=1}^{3} \left(\mu_{A_{3i}}(x) - \nu_{A_{3i}}(x)\right) e^{\pi_{A_{3i}}(x)}$$

Then, the calculation results are;

$$S_{SH}(A_1) = \frac{1}{3}\{(1 - 0) + (0.8 - 0)e^{0.2} + (0.7 - 0.1)e^{0.2}\} = 0.587$$
$$S_{SH}(A_2) = \frac{1}{3}\{(0.8 - 0.1)e^{0.1} + (1.0 - 0) + (0.9 - 0.1)\} = 0.663$$
$$S_{SH}(A_3) = \frac{1}{3}\{(0.6 - 0.2) + (0.8 - 0)e^{0.2} + (1.0 - 0)\} = 0.611$$

Calculation results illustrate the preference to the specific criterion as $A_1 \prec A_3 \prec A_2$. The results are compared with the existing score functions (Chen and Tan, 1994; Liu and Wang, 2007):

From $S_C(A) = \mu_A(x) - \nu_A(x)$, $S_C(A_1) = 0.8$, $S_C(A_2) = 0.833$, $S_C(A_3) = 0.733$. Then $A_3 \prec A_1 \prec A_2$. And with $S_L(A) = \mu_A(x) + \mu_A(x)\pi_A(x)$, $S_L(A_1) = 0.933$, $S_L(A_2) = 0.926$, $S_L(A_3) = 0.893$. Then $A_3 \prec A_2 \prec A_1$. Calculation results indicate the proposed score function $S_{SH}(A)$ match with the most preference is the same with $S_C(A)$. It is based on the difference between $\mu_A(x)$ and $\nu_A(x)$. Whereas $S_L(A)$ has the trend to the optimistic viewpoint.

4.2. Multi-criteria problem

Now we consider with the class label of C_1, C_2, and C_m for the attributes; P_1, P_2 and P_n as the function of

$$P_i = \{\langle x_1, \mu_1(x_1), \nu_1(x_1)\rangle, \langle x_2, \mu_2(x_2), \nu_2(x_2)\rangle, \dots, \langle x_n, \mu_n(x_n), \nu_n(x_n)\rangle\}. \quad (14)$$

Attributes have the pattern over an universe of discourse $X = \{x_1, x_2, \dots, x_n\}$, and they are represented by the following example on IFSs

$$P_1 = \{\langle x_1, 0.5, 0.2\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.4, 0.2\rangle, \langle x_4, 0.5, 0.3\rangle\},$$
$$P_2 = \{\langle x_1, 0.5, 0.3\rangle, \langle x_2, 0.5, 0.2\rangle, \langle x_3, 0.4, 0.2\rangle, \langle x_4, 0.3, 0.5\rangle\},$$
$$P_3 = \{\langle x_1, 0.5, 0.2\rangle, \langle x_2, 0.5, 0.1\rangle, \langle x_3, 0.4, 0.3\rangle, \langle x_4, 0.5, 0.3\rangle\}.$$

Then, the problem is designed as the 3 – criteria and 3 – attributes decision problem. But three attribute stand the sttitute to specific criterion. So it needs additional IFS value to the additional criteria as above the same. Using Boran and Akay data, $S_{SH}(A) = (\mu_A(x) - \nu_A(x))e^{\pi_A(x)}$ is calculated to each attribute;

$$S_{SH}(P_1) = \frac{1}{4}\{0.3e^{0.3} + 0.3e^{0.3} + 0.2e^{0.4} + 0.2e^{0.2}\} = 0.204,$$
$$S_{SH}(P_2) = \frac{1}{4}\{0.2e^{0.2} + 0.3e^{0.3} + 0.2e^{0.4} - 0.2e^{0.2}\} = 0.115,$$
$$S_{SH}(P_3) = \frac{1}{4}\{0.3e^{0.3} + 0.4e^{0.4} + 0.1e^{0.3} + 0.2e^{0.2}\} = 0.217.$$

Then, scoring function calculation follows as above, and it is placed in Table 1. The absolute order on all criteria is the same. With different preference in Eq. (14) for C_2 or C_3, score function could be changed.

Table 1: 3 – criteria and 3 – attributes decision problem.

Comparable Attributes	C_1	C_2	C_3
P_1	0.204	–	–
P_2	0.115	–	–
P_3	0.217	–	–

Most of all, we need preference order among criteria such as $C_1 \prec C_2 \prec C_3$ or any order. Then the total order of attributes can be derived with lexicographical order in subsection 3.3.

4.3. Analysis and discussion

To decide multi-criteria decision, there are two requirements as below;

- Criteria preference, emphasizing the priority of each criterion whether it is order or weight for each criterion.
- Score function to numerate attributes in the specific criterion. In this case, variate preference value of each attribute to the corresponding criterion.
- If each criterion has their attitude as IFS, then we can consider score function as the similarity measure between criterion and each attribute – how they are close.
- For each value in attribute compose $n \times m$ matrix, with number of attributes n and the number of criteria m.

The importance of score function is that it can makes order among attributes under the any criterion. Score function design should be considered whether it is optimistic or pessimistic viewpoint; as mentioned by (Liu and Wang, 2007): it was emphasized as optimistic. Abstention valued was added to the affirmative trend, it caould be design as vice versa.

Most of all, data unification is also necessary to analyze its scoring and preference. Due to the scoring structure, consistent data expression is necessary. For example, FSs and IFSs have different structure even FSs is considered as $\pi(x) = 0$. So, the existing researes were focussed on the specific data structure, not the general.

Many researches on the similarity measure provided useful foundation to provide the closedness between comparable data sets (Lee et al., 2009; Li and Cheng, 2002; Li et al., 2007; Liu, 1992). It is also possible to apply the result to the multi-criteria decision problem, but criteria aso need to has intuitionistic values in advance. Because of similarity measure structure, compatible data can be possible to calculate similarity measure.

5. CONCLUSIONS

Score function has been proposed by using hesitation analysis on the intuitionistic fuzzy sets. Hesitation shows abstention in the middle of decision making; it can be included in any affirmative or dissentient attitude. Even the score function design is close to surjective viewpoint, it should be close to the rationality and consistency.

By deriving the score function, recent hesitation analysis results are shared with negative and positive hesitation. Classifying with two area depend on whether the data are overlapped or non-overlapped with respect to the comparable data sets. Together with the analysis and discussion on the existing research, we proposed the score function by considering membership and non-membership degree difference. In the structure, it multiplies the hesitation natural power value, it means equivalent weights are added to the affirmative and dissentient tendency.

To review the decision result, single criterion and multi-criteria examples are illustrated. In this research, we score the order inside of attributes under the single criterion problem. However, criteria preference order is needed under the multi-criteria problem. So, the lexicographical knowledges are needed to implement the total order – dictionary order is also useful with Cartesian product structure. It was summarized slightly in subsection 3.3.

By the consideration of hesitation, we proposed novel score function. It has the structure including the difference of affirmative and dissentient together with hesitation natural power in natural number. The structure avoids the problem in Chen and Tan's score function – same difference between $\mu(x)$ and $\nu(x)$ even they have different calue each other. Hence, the research output could be the foundation to develop more effective score function in future.

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